Earth's Size

- What are the early ideas on the Earth's shape?
  - What are the qualitative observations against them?
- Understand how Eratosthenes measured the Earth's size.
  - His estimate of the Earth's circumference is 46,250 km. 15 % greater than the modern value, 40,030 km.
- A French astronomer, Picard measured the length of one degree of meridian arc in 1671. From it, he got 6372 km as the radius of the Earth: cf. modern value, 6371 km Hypothetical mountain belts

Earth's Shape

- Jean Richer's finding (1672) that his accurate pendulum clock, calibrated in Paris, was "losing 2m 30s" (i.e., indicated an increased period of the pendulum!) on an equatorial island of Cayenne.

Earth's Shape

- Jean Richer's finding (1672) cont'd.
  - What does the increased period by 2m 30s imply about the shape of the Earth?
  - Hint: $T = 2\pi\sqrt{\frac{l}{g}}$
- What was Newton's idea on the shape of the Earth? Oblate or prolate?
- How was the length of a degree of meridian measured?

Earth's Gravity

- Newton's law of universal gravitation:

$$\mathbf{F} = -G\frac{m\,M}{r^2}\hat{\mathbf{r}}$$

  - Gravitational acceleration

$$\mathbf{a}_G = -G\frac{M}{r^2}\hat{\mathbf{r}}$$

  - Gravitational potential

$$U_G = -G\frac{M}{r}$$

- How to measure $\mathbf{a}_G$?
- What is the mass and the mean density of the Earth?

Earth's Gravity

- How to measure G?
  - Cavendish's experiment with metal balls (1798).
  - see https://www.aps.org/publications/apsnews/201605/big-g.cfm
  - The current best estimate for G is $6.67408 \pm 0.00031 \times 10^{-11}$ $m^3 kg^{-1} s^{-2}$ (CODATA-2014[1]).

[1] Mohr, P. J., Newell, D. B., and Taylor, B. N. 2016. CODATA recommended values of the fundamental physical constants: 2014. Rev. Mod. Phys., 88, 1–71.

Earth's Gravity

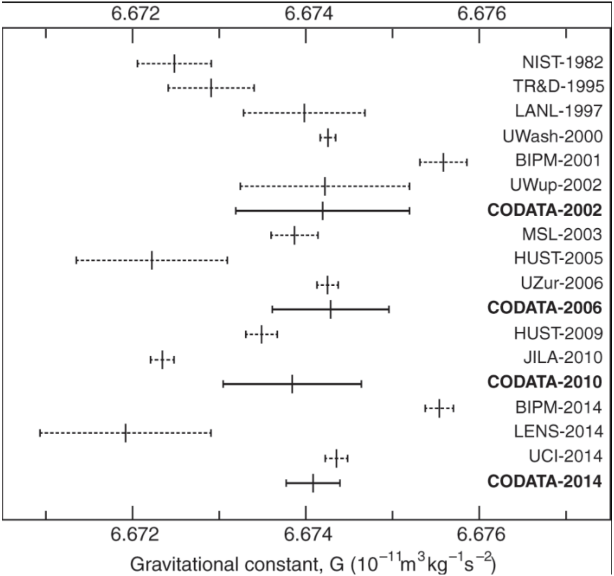


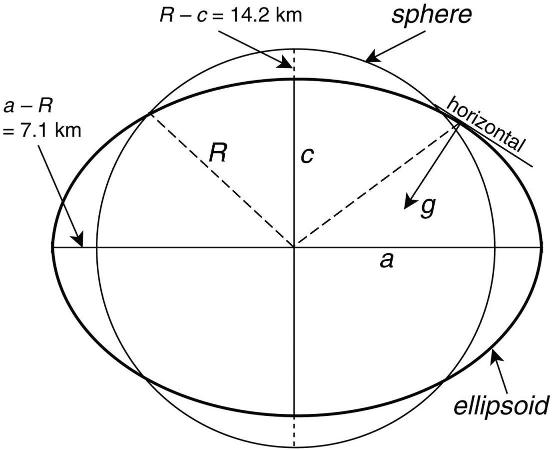
Fig. B3.1.1 (Lowry et al., 2020)

Earth's Figure and Gravity[2]

- The figure of the Earth: *Approximated* as the shape of an equipotential surface of gravity that fits the mean sea level best. It's called the **geoid**.
- The best *mathematical approximation* to the figure is an oblate ellipsoid or spheroid called the *International Reference Ellipsoid*, or just the **reference ellipsoid**. Its specs are defined in Geodetic Reference System 1980 (GRS80).
  - Equatorial radius (a) = 6378.137 km,
  - polar radius (c) = 6356.752 km,
  - Radius of the equivalent sphere (R) = $(a^2c)^{1/3}$ = 6371.000 km.
  - Same with the reference ellipsoid defined in the WGS (World Geodetic System) 84 standard although c of WGS 84 is greater by 0.105 mm.

[2] Sec. 2.4 of (Lowry, 2007)

Earth's Figure and Gravity (cont'd)



$a = 6378.137$ km
$c = 6356.752$ km
$R = 6371.001$ km

(FoG, 3rd ed., 2020)

Earth's Figure and Gravity (cont'd)

- The polar **flattening**, $f = (a - c)/a$
  - f_{ref} of the reference ellipsoid = 1/298.252 = 3.35287×10^{-3}
  - f_{hyd} of the rotating fluid in hydrostatic equilibrium = 1/299.5.
  - What does "$f_{ref} > f_{hyd}$" imply?
- **Dynamical ellipticity**, $H = (C - A)/C$, where C is the moment of inertia of the Earth about the rotation axis and A about any equatorial axis.
  - $H = 3.27379 \times 10^{-3}$ = 1/305.456

Earth's Figure and Gravity (cont'd)

- For later convenience, let's also define μ, the ratio of the equatorial centrifugal acceleration to the equatorial gravity:

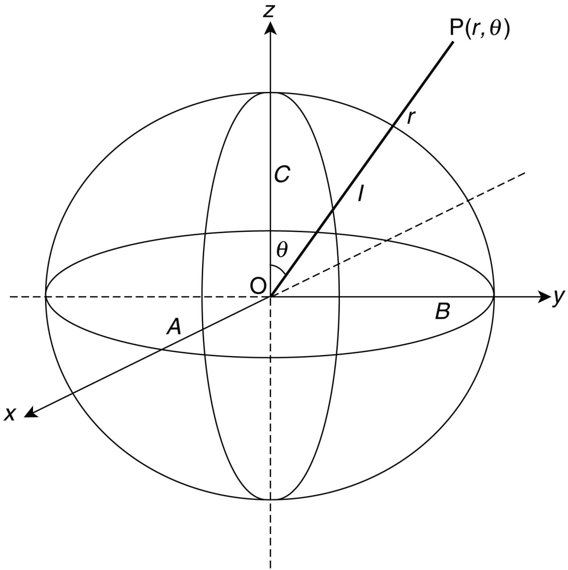
$$\mu = \frac{\omega^2 a}{g_e}$$

- McCullagh's formula: Describes the most important two contributions to the gravitational potential of any body:

$$U_G = -G\frac{E}{r} - G\frac{(A + B + C - 3\,I)}{2\,r^3}.$$

- A review of moment of inertia of a rigid body can be found at the end.

Earth's Figure and Gravity (cont'd)



(FoG, 3rd ed., 2020)

Earth's Figure and Gravity (cont'd)

- For the *spheroidal* (i.e., oblate ellipsoidal) Earth, $A = B$ and also $I = A \sin^2\theta + C \cos^2\theta$.

$$U_G = -G\frac{E}{r} + G\frac{(C-A)}{r^3}\frac{(3\cos^2\theta - 1)}{2}$$

- The 2nd term can be expressed in terms of the Legendre polynomial[3]:

$$U_G = -G\frac{E}{r} + G\frac{(C-A)}{r^3}P_2(\cos\theta)$$

- By rearranging terms, we get

$$U_G = -G\frac{E}{r}\left[1 - \left(\frac{C-A}{ER^2}\right)\left(\frac{R}{r}\right)^2 P_2(\cos\theta)\right]$$

[3] See Box 2.2 of Lowry (2007).

Earth's Figure and Gravity (cont'd)

- Represented by the Legendre polynomials (cf. *spherical harmonics*[4]), the potential field for a spheroid can be expressed as

$$U_G = -G\frac{E}{r}\left[1 - \sum_{n=2}^{n=\infty}\left(\frac{R}{r}\right)^n J_n P_n(\cos\theta)\right]$$

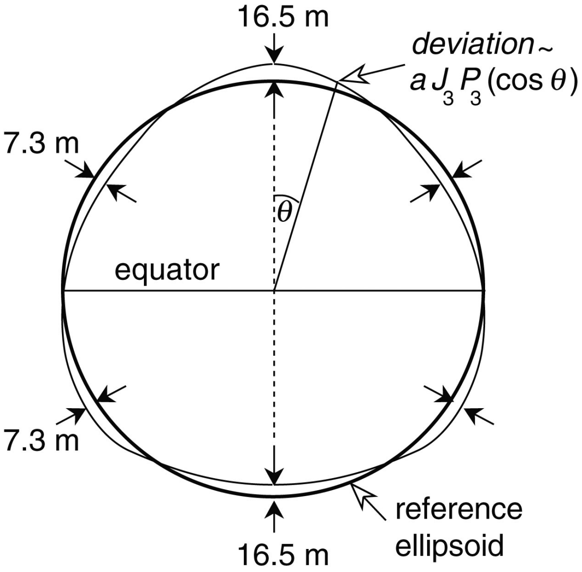
- Comparing this equation with the MacCullagh's formula term by term, we find

$$J_2 = \frac{C - A}{ER^2}$$

- Higher order terms (e.g., n=3) represent deviations from the ellipsoid, which correspond to a pear-shaped Earth.

[4] See Box 2.3 of Lowry (2007) and also http://scipython.com/book/chapter-8-scipy/examples/visualizing-the-spherical-harmonics/

Earth's Figure and Gravity (cont'd)



(FoG, 3rd ed., 2020)

Earth's Figure and Gravity (cont'd)

- **Geopotential**: The sum of the gravitational and centrifugal potentials. On a rotating spheroid,

$$U_g = U_G - \frac{1}{2}\omega^2 r^2 \sin^2\theta$$

- From the definitions of f, μ (equatorial centrifugal to grav. acc.), we find

$$J_2 = \frac{1}{3}(2f - \mu)$$

- Equating this with $J_2 = (C - A)/(ER^2)$, we get

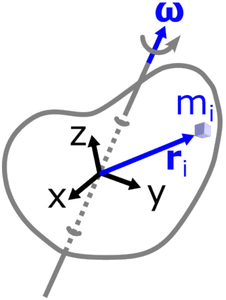
$$\frac{C - A}{C}\frac{C}{ER^2} = \frac{1}{3}(2f - \mu)$$

- $C \sim 0.33ER^2$ for the Earth.
- Noting that a hollow spherical shell has 0.66 ER^2 and a solid sphere 0.4ER^2, can you guess what the Earth's value mean?

On moment of inertia

Some notes on angular momentum and moment of inertia

	Linear momentum	Angular momentum
Symbol	$\mathbf{p}$	$\mathbf{L}$
Definition	m $\mathbf{v}$	$\mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v})$
Unit	kg m/s	kg m^2/s
Balance law	$\mathbf{p} =$m $\mathbf{v}$	$\mathbf{L} = \mathbf{I}\,\omega$



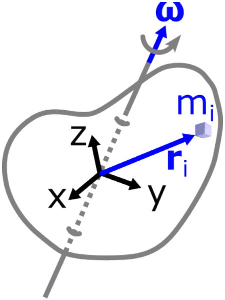
$$\mathbf{L} = \sum_i m_i(\mathbf{r}_i \times \mathbf{v}_i) = \sum_i m_i\,[\mathbf{r}_i \times (\omega \times \mathbf{r}_i)]$$

$$= \sum_i m_i \left[\omega r_i^2 - \mathbf{r}_i(\mathbf{r}_i \cdot \omega)\right]$$

Here, we used the following vector identity:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

On moment of inertia



$$L_x = \sum_i \left[m_i \, \omega_x r_i^2 - x_i(\omega_x x_i + \omega_y y_i + \omega_z z_i) \right]$$

$$= \left(\sum_i m_i (r_i^2 - x_i^2) \right) \omega_x + \left(-\sum_i m_i x_i y_i \right) \omega_y$$

$$+ \left(-\sum_i m_i x_i z_i \right) \omega_z$$

$$= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

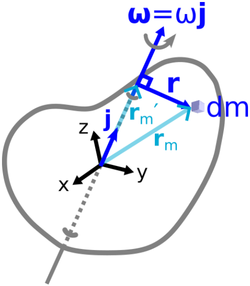
$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

$$\Rightarrow \mathbf{L} = \mathbf{I}\boldsymbol{\omega},$$

where **I** is the moment of inertia tensor that is symmetric positive definite (i.e., diagonalizable).

On moment of inertia



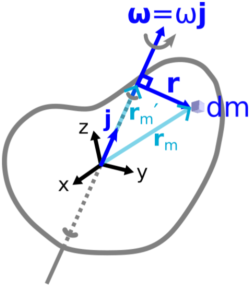
Note the following property of *direction cosines*: For a given vector $\mathbf{r}$,

$$\mathbf{r}\cdot\mathbf{e}_x = |\mathbf{r}||\mathbf{e}_x|\cos\alpha \rightarrow \cos\alpha = \frac{\mathbf{r}\cdot\mathbf{e}_x}{|\mathbf{r}|}.$$

Likewise, $\cos\beta = \mathbf{r}\cdot\mathbf{e}_y/|\mathbf{r}|$ and $\cos\gamma = \mathbf{r}\cdot\mathbf{e}_z/|\mathbf{r}|$.

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

On moment of inertia



$$|r'_m| = \mathbf{r}_m \cdot \hat{\mathbf{j}} = (x, y, z) \cdot (\cos\alpha, \cos\beta, \cos\gamma) = x\cos\alpha + y\cos\beta + z\cos\gamma.$$

$$I = \int r^2 dm = \int (r_m^2 - r_m'^2) dm = \int \left[(x^2 + y^2 + z^2) - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2 \right] dm$$

$$I = \int (1 - \cos^2\alpha) x^2 dm + \int (1 - \cos^2\beta) y^2 dm + \int (1 - \cos^2\gamma) z^2 dm + \text{ other terms}$$

On moment of inertia

Using the property of direction cosines, we get

$$
\begin{aligned}
I &= (\cos^2\beta + \cos^2\gamma)\int x^2 dm + (\cos^2\alpha + \cos^2\gamma)\int y^2 dm \\
&\quad + (\cos^2\alpha + \cos^2\beta)\int z^2 dm + \text{ other terms} \\
&= \cos^2\alpha \int (y^2+z^2) dm + \cos^2\beta \int (x^2+z^2) dm \\
&\quad + \cos^2\gamma \int (x^2+y^2) dm + \text{ other terms} \\
&= \cos^2\alpha\, I_x + \cos^2\beta\, I_y + \cos^2\gamma\, I_z + \text{ other terms} .
\end{aligned}
$$

If the principal axes are chosen,

$$
I = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma.
$$

On moment of inertia

- An interesting example of moment of inertia optimization
  - https://youtu.be/qquek0c5bt4
  - Theory paper: https://la.disneyresearch.com/publication/spin-it-optimizing-moment-of-inertia-for-spinnable-objects/