Kevorkian, J.
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Analytical Solution Techniques
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The Diffusion Equation

In this chapter we study the diffusion equation

$$u_t - (u_{xx} + u_{yy} + u_{zz}) = p(x, y, z, t),$$

which describes a number of physical models, such as the conduction of heat in a solid or the spread of a contaminant in a stationary medium.

We shall use this equation to introduce many of the solution techniques that will be useful in subsequent chapters in our study of other types of linear partial differential equations. To begin with, it is important to have a physical understanding of how the diffusion equation arises in a particular application, and we consider the simple model of heat conduction in a solid.

1.1 Heat Conduction

Consider a thin axisymmetric rod of some heat-conducting material with variable density $\rho(x)$ (g/cm^3) (for example, a copper–silver alloy with a variable copper/silver ratio along the rod). Let $A(x)$ (cm^2) denote the cross-sectional area and assume that the surface of the rod is perfectly insulated so that no heat is lost or gained through this surface. (See Figure 1.1.) Thus, the problem is one-dimensional in the sense that all material properties depend on the distance x along the rod. We assume that at each spatial position x and time t there is one temperature θ that does not depend on the transverse coordinates y or z. Let x_1 and x_2 be two arbitrary fixed points on the axis.

In the basic law of conservation of heat energy for the rod segment $x_1 \le x \le x_2$, the rate of change of heat inside this segment is equal to the net flow of heat through the two boundaries at x_1 and x_2, plus the heat produced by a possible distribution of internal heat sources in the interval. Consider an infinitesimal section of length dx in the interval $x_1 \le x \le x_2$. Using elementary physics, we have dQ, the heat content in this section, proportional to the mass and the temperature:

$$dQ \equiv c(\rho A dx)\theta, \tag{1.1.1}$$

where the constant of proportionality c is the specific heat in cal/g°C. Thus, the total heat content in the interval $x_1 \leq x \leq x_2$ is*

$$Q(t) \equiv \int_{x_1}^{x_2} c(x)\rho(x)A(x)\theta(x,t)dx. \tag{1.1.2}$$

Next, we invoke Fourier's law for heat conduction, which states that the rate of heat flowing *into* a body through a small surface element on its boundary is proportional to the area of that element and to the *outward* normal derivative of the temperature at that location. The constant of proportionality here is $k \sim$ (cal/cm s°C), the *thermal conductivity*. Note that this sign convention implies the intuitively obvious fact that the direction of heat flow between two neighboring points is toward the relatively cooler point. For example, if the temperature increases as a boundary point is approached from inside a body, then the outward normal derivative of the temperature is positive, and this correctly implies that heat flows into the body.

For the present one-dimensional example, the net inflow of heat through the boundaries x_1 and x_2 is

$$R(t) \equiv A(x_2)k(x_2)\frac{\partial\theta}{\partial x}(x_2,t) - A(x_1)k(x_1)\frac{\partial\theta}{\partial x}(x_1,t). \tag{1.1.3}$$

Let $h(x,t)$ (cal/g s) denote the heat produced per unit mass and time by the sources. Thus, the total time rate of heat production by the sources is

$$H(t) \equiv \int_{x_1}^{x_2} h(x,t)\rho(x)A(x)dx. \tag{1.1.4}$$

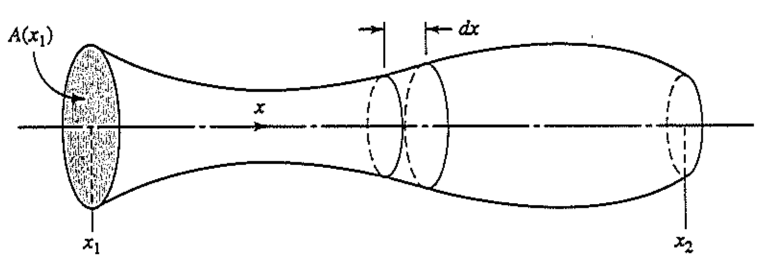

FIGURE 1.1. Thin axisymmetric heat conductor

* In this text we shall often use the notation $\equiv$ instead of $=$ when it is important to indicate that a new quantity is being defined, as in (1.1.1) and (1.1.2). As a special case of this notation, the statement $f(x,y) \equiv 0$ indicates that the function f of x and y vanishes identically; that is, it equals zero for all x and y by definition.

The conservation of heat then implies

$$\frac{dQ}{dt} = R(t) + H(t), \tag{1.1.5}$$

or

$$\frac{d}{dt}\int_{x_1}^{x_2} c(x)\rho(x)A(x)\theta(x,t)dx = A(x_2)k(x_2)\frac{\partial\theta}{\partial x}(x_2,t) - A(x_1)k(x_1)\frac{\partial\theta}{\partial x}(x_1,t) + \int_{x_1}^{x_2} h(x,t)\rho(x)A(x)dx. \tag{1.1.6}$$

Equation (1.1.6) is a typical integral *conservation law*, which has general applicability. For example, (1.1.6) remains true if material properties have a discontinuity at a given point $x = \xi$ inside the interval, as would be the case if we had a perfect thermal bond between two rods of different materials. We shall encounter other examples of such conservation laws later on in the book and shall study how discontinuities propagate in detail in Chapter 5.

For smooth material properties, that is, if c, ρ, A, and k are continuous and have a continuous first derivative, the solution $\theta(x,t)$ is also continuous with continuous first partial derivatives $\partial\theta/\partial x$ and $\partial\theta/\partial t$, and we may rewrite (1.1.6) in the following form after we express $R(t)$ as the integral of a derivative:

$$\int_{x_1}^{x_2}\left\{c(x)\rho(x)A(x)\frac{\partial\theta}{\partial t}(x,t) - \frac{\partial}{\partial x}\left[A(x)k(x)\frac{\partial\theta}{\partial x}(x,t)\right] - h(x,t)\rho(x)A(x)\right\}dx = 0. \tag{1.1.7}$$

Since (1.1.7) is true for any x_1 and x_2, it follows that the integrand must vanish:

$$c(x)\rho(x)A(x)\frac{\partial\theta}{\partial t} - \frac{\partial}{\partial x}\left[A(x)k(x)\frac{\partial\theta}{\partial x}\right] = h(x,t)\rho(x)A(x). \tag{1.1.8}$$

For constant area and material properties, this reduces to

$$\frac{\partial\theta}{\partial t} - \kappa^2\frac{\partial^2\theta}{\partial x^2} = \sigma(x,t), \tag{1.1.9}$$

where $\kappa^2 \equiv k/c\rho$ (cm²/s) is the *thermal diffusivity* and $\sigma \equiv h/c$. The dimensionless form of (1.1.9) follows when characteristic constants with dimensions of temperature, length, and time are used to define nondimensional variables.

For example, let us study (1.1.9) for a rod of length L that is initially at a constant temperature θ_0 and has one end, $x = L$, held at $\theta = \theta_0$ while the other end, $x = 0$, has a prescribed temperature history $\theta(0,t) = \theta_0 f(t/T)$, where T is a characteristic time scale. For simplicity assume $\sigma \equiv 0$. We set

$$u \equiv \frac{\theta}{\theta_0}, \quad x^* \equiv \frac{x}{L}, \quad t^* \equiv \frac{t\kappa^2}{L^2},$$

and obtain the following dimensionless formulation:

$$\frac{\partial u}{\partial t^*} - \frac{\partial^2 u}{\partial x^{*2}} = 0, \quad 0 \leq x^* \leq 1, \tag{1.1.10a}$$

$$u(x^*, 0) = 1 \tag{1.1.10b}$$

$$u(0, t^*) = f(\lambda t^*), \quad t^* > 0, \tag{1.1.10c}$$

$$u(1, t^*) = 1, \tag{1.1.10d}$$

where λ is the dimensionless parameter $L^2/(\kappa^2 T)$. The original dimensional formulation of this problem involves the four constants κ, θ_0, L, and T. The dimensionless description is considerably simpler, as it involves only the one parameter λ. Once the dimensionless problem has been solved, say $u = U(x^*, t^*)$, the dimensional result is easily obtained in the form

$$\theta = \theta_0 U\left(\frac{x}{L}, \frac{t\kappa^2}{L^2}\right).$$

The corresponding derivation for three-dimensional heat conduction follows from similar steps. If a solid occupies the domain G with surface S and outward unit normal $\mathbf{n}$, as shown in Figure 1.2, the total heat content of the solid is given by

$$Q(t) \equiv \iiint_G c\rho\theta \, dV, \tag{1.1.11}$$

where dV is the volume element; for instance, $dV = dx\,dy\,dz$ in Cartesian variables. The net *inflow* of heat through the boundary S is

$$R(t) \equiv \iint_S k \operatorname{grad} \theta \cdot \mathbf{n} \, dA. \tag{1.1.12}$$

We can express $R(t)$ in terms of a volume integral over G using Gauss' theorem. This theorem states that if $\mathbf{F}$ is a one-valued vector field with continuous first partial

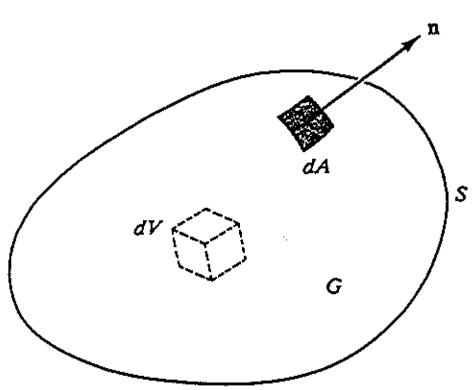

FIGURE 1.2. Three-dimensional heat conductor

derivatives in G, then*

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iiint_G \operatorname{div} \mathbf{F} dV. \tag{1.1.13}$$

Therefore, for a medium where c, ρ, and k are smooth, we identify $\mathbf{F}$ in (1.1.13) with $k \operatorname{grad} \theta$, and (1.1.12) becomes

$$R(t) = \iiint_G \operatorname{div}(k \operatorname{grad} \theta) dV. \tag{1.1.14}$$

Also, since G is fixed in space we have

$$\frac{dQ}{dt} \equiv \frac{d}{dt} \iiint_G c\rho\theta dV = \iiint_G c\rho\theta_t dV. \tag{1.1.15}$$

The conservation law of heat energy (1.1.5) becomes

$$\iiint_G c\rho\theta_t dV = \iiint_G \operatorname{div}(k \operatorname{grad} \theta) dV + \iiint_G h\rho dV. \tag{1.1.16}$$

Therefore, assuming continuity of the integrands in (1.1.16), the three-dimensional version of (1.1.8) is

$$c\rho\theta_t - \operatorname{div}(k \operatorname{grad} \theta) = h\rho. \tag{1.1.17}$$

For constant k, this reduces to

$$\theta_t - \kappa^2 \Delta\theta = \sigma, \tag{1.1.18}$$

where $\kappa^2 \equiv k/(c\rho)$, $\sigma \equiv h/c$, and Δ is the Laplacian operator $\Delta \equiv \operatorname{div} \operatorname{grad}$, given by

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.1.19}$$

in Cartesian coordinates.

1.2 The Fundamental Solution

The *fundamental solution* of a second-order partial differential equation is just Green's function for that equation over the infinite domain with zero boundary conditions (if appropriate) at infinity. See Appendix A.1 for a review of the use of

* Thus, in writing $R(t)$ in the form given in (1.1.7), we have used the "one-dimensional version" of Gauss' theorem relating the definite integral of the derivative of a function to values of the function at the endpoints.

Green's function in ordinary differential equations. For example, the fundamental solution of the one-dimensional diffusion equation obeys

$$u_t - u_{xx} = \delta(x-\xi)\delta(t-\tau) \tag{1.2.1}$$

on $-\infty < x < \infty$, $0 \le t < \infty$, where ξ and τ are fixed constants, $|\xi| < \infty$, $0 \le \tau < \infty$, and δ denotes the Dirac delta function. We may interpret (1.2.1) physically as the equation governing the temperature in an infinite conductor that is subjected to a *concentrated unit source of heat* at the point $x = \xi$. This source of heat is turned on only for the "instant" $t = \tau$ and is absent for all other times; its location is also concentrated at the point $x = \xi$.

Prior to the application of the heat source, the conductor has a constant temperature that we normalize to equal zero. Thus, the boundary conditions are

$$u(x,t) \to 0 \quad \text{as } |x| \to \infty, \tag{1.2.2}$$

and the initial condition is

$$u(x,t) = 0; \quad 0 \le t < \tau. \tag{1.2.3}$$

The solution of (1.2.1)–(1.2.3) is the fundamental solution, which is a function of $x - \xi$ and $t - \tau$,

$$u = F(x-\xi, t-\tau). \tag{1.2.4}$$

There is no loss of generality in taking the initial and boundary temperatures equal to zero in (1.2.2)–(1.2.3); any constant value u_0 can be used and then reduced to (1.2.2)–(1.2.3) by simply considering the new dependent variable $u - u_0$. This is a consequence of the absence of nondifferentiated terms in (1.2.1). Also, since the left-hand side of (1.2.1) does not involve x or t, we need only consider the simpler problem corresponding to $\xi = \tau = 0$

$$u_t - u_{xx} = \delta(x)\delta(t), \tag{1.2.5}$$

$$u(x, 0^-) = 0, \tag{1.2.6}$$

$$u(x,t) \to 0 \quad \text{as } |x| \to \infty. \tag{1.2.7}$$

Once the solution $u = F(x,t)$ of (1.2.5)–(1.2.7) is found, the general result $F(x-\xi, t-\tau)$ is obtained by translation.

In Section 1.3 we shall show that once the fundamental solution is known, we can solve the following *general* initial-value problem for the diffusion equation on the infinite domain

$$u_t - u_{xx} = p(x,t); \quad -\infty < x < \infty; \quad 0 \le t < \infty, \tag{1.2.8}$$

$$u(x,0) = f(x), \tag{1.2.9}$$

$$u(x,t) \to f(\pm\infty) \quad \text{as } x \to \pm\infty, \tag{1.2.10}$$

where p and f are prescribed functions and $p(x,t) \equiv 0$ if $t < 0$.

In the next three subsections we derive the fundamental solution F using different techniques that have a broad range of applicability in solving partial differential equations.

1.2.1 Similarity (Invariance)

In this very useful approach, we ask under what scalings of the dependent and independent variables the system (1.2.5)–(1.2.7) is invariant. If such scalings exist, we can reduce (1.2.5) to an ordinary differential equation in terms of a "similarity" variable using arguments that go as follows.

Assume that we have found the solution of (1.2.5)–(1.2.7) in the form $u = F(x,t)$. Is it possible to use this result to obtain a second solution $u = G(x,t)$ by setting $\bar{x} = \beta x$ and $\bar{t} = \gamma t$ and defining G by

$$G(x,t) \equiv \alpha F(\beta x, \gamma t) \tag{1.2.11}$$

for positive constants α, β, and γ?

We compute $G_t = \alpha\gamma F_{\bar{t}}$, $G_{xx} = \alpha\beta^2 F_{\bar{x}\bar{x}}$, and use of the fact that for any constant c, we may set (See (A.1.16))

$$\delta(cx) \to \frac{1}{|c|}\delta(x). \tag{1.2.12}$$

If $G(x,t)$ is to be a solution of (1.2.5)–(1.2.7), we must have

$$G_t - G_{xx} = \delta(x)\delta(t), \quad G(x,0^-) = 0, \quad G(x,t) \to 0 \quad \text{as } |x| \to \infty. \tag{1.2.13}$$

Expressing G_t and G_{xx} in terms of $F_{\bar{t}}$ and $F_{\bar{x}\bar{x}}$ and using $\delta(x)\delta(t) = \delta(\bar{x}/\beta)\delta(\bar{t}/\gamma) = \beta\gamma\delta(\bar{x})\delta(\bar{t})$ in (1.2.13) gives

$$\alpha\gamma F_{\bar{t}} - \alpha\beta^2 F_{\bar{x}\bar{x}} = \beta\gamma\delta(\bar{x})\delta(\bar{t}),$$
$$\alpha F(\bar{x}, 0^-) = 0, \quad \alpha F(\bar{x}, \bar{t}) \to 0 \quad \text{as } |x| \to \infty,$$

or

$$F_{\bar{t}} - \left(\frac{\beta^2}{\gamma}\right) F_{\bar{x}\bar{x}} = \left(\frac{\beta}{\alpha}\right)\delta(\bar{x})\delta(\bar{t}),$$
$$F(\bar{x}, 0^-) = 0, \quad F(\bar{x}, \bar{t}) \to 0 \quad \text{as } |x| \to \infty.$$

But we know that $F(\bar{x}, \bar{t})$ must satisfy (1.2.5)–(1.2.7) in terms of the $\bar{x}, \bar{t}$ variables. Therefore, $G(x,t)$, as defined by (1.2.11), can be a solution only if $\beta^2/\gamma = 1$ and $\beta/\alpha = 1$; that is, if $\beta = \alpha$ and $\gamma = \alpha^2$. Thus, (1.2.11) must be of the form

$$G(x,t) = \alpha F(\alpha x, \alpha^2 t). \tag{1.2.14}$$

Have we discovered a new solution of (1.2.5)–(1.2.7)? Of course not; the solution for this problem is unique, $G = F$, as is physically obvious and can be proved. Therefore, (1.2.14) is just a statement of the *similarity structure* of the solution F, and (1.2.14) must read

$$\alpha F(\alpha x, \alpha^2 t) = F(x,t). \tag{1.2.15}$$

That is to say, if we replace x by αx and t by $\alpha^2 t$ in F and then multiply the result by α (for any $\alpha > 0$), *the resulting expression is identical to* $F(x,t)$. This property implies that $F(x,t)$ must be of the form

$$F(x,t) = \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right), \quad \text{or } \frac{1}{\sqrt{t}} g\left(\frac{x^2}{t}\right), \quad \text{or } \frac{1}{x} h\left(\frac{x}{\sqrt{t}}\right), \ldots$$

for certain functions $f, g, h, \ldots$ of the indicated arguments.

Any one of an infinite number of possibilities that satisfy the similarity condition (1.2.15) may be used. Each choice will reduce (1.2.5) to an ordinary differential equation, which, when solved, will give the *same* result for F. Let us pick the form

$$F(x,t) = \frac{1}{\sqrt{t}} f(\zeta), \quad \zeta \equiv \frac{x}{\sqrt{t}}.$$

We compute

$$F_x = \frac{1}{t} f'; \quad F_{xx} = \frac{1}{t^{3/2}} f''; \quad F_t = -\frac{1}{2t^{3/2}} f - \frac{x}{2t^2} f',$$

where $' \equiv d/d\zeta$.

Since the delta function on the right-hand side of (1.2.5) is identically equal to zero for $t > 0$, we need to solve only the homogeneous diffusion equation for $t > 0$. However, the initial condition $u(x, 0^-) = 0$ in (1.2.6) does not remain valid for $t = 0^+$. (If it did, the result would be the trivial solution $u(x,t) \equiv 0$.) The effect of the delta function on the right-hand side is to generate impulsively a nonzero value for $u(x, 0^+)$ (see (1.2.22)), which is the appropriate initial condition to be used in solving the homogeneous equation (1.2.5) for $t > 0$.

Consider now the homogeneous version of (1.2.5). Using the results we computed for F and its derivatives gives

$$-\frac{1}{2t^{3/2}} f - \frac{x}{2t^2} f' - \frac{1}{t^{3/2}} f'' = 0,$$

which is the linear second-order ordinary differential equation

$$f'' + \frac{\zeta}{2} f' + \frac{1}{2} f = 0 \tag{1.2.16}$$

with the independent variable ζ.

Integrating once gives $f' + (\zeta/2) f = A =$ constant, and the solution of this is

$$f = A e^{-\zeta^2/4} \int^{\zeta} e^{s^2/4} ds + B e^{-\zeta^2/4}, \quad B = \text{constant}.$$

The constants A and B are determined by considering the total heat content $H(t)$ in the bar. In terms of our dimensionless units, the total heat is just the integral of the temperature:

$$\begin{aligned} H(t) &\equiv \int_{-\infty}^{\infty} F(x,t)dx \\ &= \frac{A}{\sqrt{t}} \int_{-\infty}^{\infty} f_1\left(\frac{x}{\sqrt{t}}\right) dx + \frac{B}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-x^2/4t} dx, \end{aligned} \tag{1.2.17}$$

where we have defined

$$f_1(\zeta) \equiv e^{-\zeta^2/4} \int^{\zeta} e^{s^2/4} ds = e^{-\zeta^2/4} \int^{\zeta^2/4} e^{\sigma} \sigma^{-1/2} d\sigma.$$

Integrating the second expression for f_1 by parts shows that (see Section A.3.5)

$$f_1(\zeta) = \frac{2}{|\zeta|} + O(\zeta^{-3}) \quad \text{as } |\zeta| \to \infty.$$

Therefore, $(1/\sqrt{t}) \int_{-\infty}^{\infty} f_1\, dx$ in (1.2.17) is unbounded. Since the total heat must be finite, we set $A = 0$ and have

$$F = \frac{B}{\sqrt{t}} e^{-x^2/4t}, \quad t > 0. \tag{1.2.18}$$

The idea now is to pick B in order to satisfy (1.2.5) at $t = 0^+$. If we differentiate the integral defining $H(t)$ in (1.2.17) with respect to t and use (1.2.5), we obtain

$$\frac{dH}{dt} = \int_{-\infty}^{\infty} F_t(x,t)dx = \int_{-\infty}^{\infty} [F_{xx}(x,t) + \delta(x)\delta(t)]dx,$$

so that

$$\frac{dH}{dt} = F_x(\infty, t) - F_x(-\infty, t) + \delta(t) = \delta(t),$$

because the temperature gradient at $\pm\infty$ due to a unit source must be zero. Therefore, $H(t)$ is the Heaviside function (see (A.1.14)), and for $t > 0$, we have

$$1 = \int_{-\infty}^{\infty} \frac{B}{\sqrt{t}} e^{-x^2/4t} dx. \tag{1.2.19}$$

Thus, after switching on a unit source of heat for an instant at the origin, the total heat content in the rod remains constant, and this constant can be set equal to unity under an appropriate nondimensionalization.

We can rewrite (1.2.19) as

$$1 = 2B \int_{-\infty}^{\infty} \frac{e^{-x^2/4t}}{\sqrt{4t}} dx = 2B \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = 2B\sqrt{\pi},$$

or

$$B = \frac{1}{2\sqrt{\pi}},$$

and the fundamental solution is

$$F(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}. \tag{1.2.20}$$

More generally, the solution of (1.2.1)–(1.2.3) is

$$F(x-\xi, t-\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} e^{-(x-\xi)^2/4(t-\tau)}. \tag{1.2.21}$$

It is important to note that the use of similarity is not restricted to linear problems. For example, a classical use of similarity arguments is provided by the boundary-layer equations for viscous incompressible flow over an infinite wedge (or the special case of a semi-infinite flat plate if the wedge angle is zero). See Section B.14 of [31]. Here, the nonlinear partial differential equation for the flow stream function is reduced to a third-order nonlinear ordinary differential equation.

A crucial requirement for the applicability of similarity arguments is that both the governing equations *and* initial and/or boundary conditions be reducible to similarity form. In the preceding example, this was trivially true for the given initial condition $F = 0$, as this also immediately implied $G = 0$. For further reading on similarity methods, see [6] and [41].

The fundamental solution (1.2.20) can also be derived using Fourier or Laplace transforms. A review of these techniques appears in Appendix 2, where this problem is used as one of the illustrative examples.

1.2.2 Qualitative behavior; diffusion

Figure 1.3 shows three temperature profiles for $F(x, t)$ given by (1.2.20) taken at three successive times $0 < t_1 < t_2 < t_3$. In each case, the area under the curve is, according to (1.2.19), equal to unity. For t smaller and smaller, the contribution to this area becomes more and more concentrated at the origin. This is just one of the many possible representations of the delta function (for instance, see (A.1.11b) with $\Delta\xi = 4t$), and we may write

$$F(x, 0^+) = \delta(x). \tag{1.2.22}$$

Equation (1.2.22) also follows by integrating (1.2.5) with respect to t from $t = 0^-$ to $t = 0^+$ and noting that $\int_{0^-}^{0^+} u_{xx} dt = 0$.

The fundamental solution can be used to give a precise definition of *diffusion*. First, notice that if we regard the source at $x = 0$ as a disturbance introduced at time $t = 0$, the "signal speed" due to this disturbance is *infinite* because for any positive t, no matter how small, the value of u is nonzero for all x. Thus, the entire rod instantly "feels" the effect of the source. Of course, a real temperature gauge would fail to detect the very weak disturbance at large distances. Thus, the idea of a signal speed is not very useful in this case, and we would like to have a better characterization of how the rod "heats up" for $t > 0$. Suppose we ask instead where a given fraction of the *total heat* in the rod is to be found at any specified time. We know that at $t = 0^+$, all the heat is concentrated at the origin. For any $t > 0$, the heat is nonuniformly distributed over the entire rod with the maximum temperature at the origin, as shown in Figure 1.4.

Suppose that d is a fixed constant with $0 < d < 1$. At some time $t > 0$, the temperature distribution is the even function of x given by (1.2.20) and sketched in Figure 1.4. The shaded area represents the fraction d of the total area (which equals unity). Thus, as t increases, so does x_d. The question is, how does x_d depend

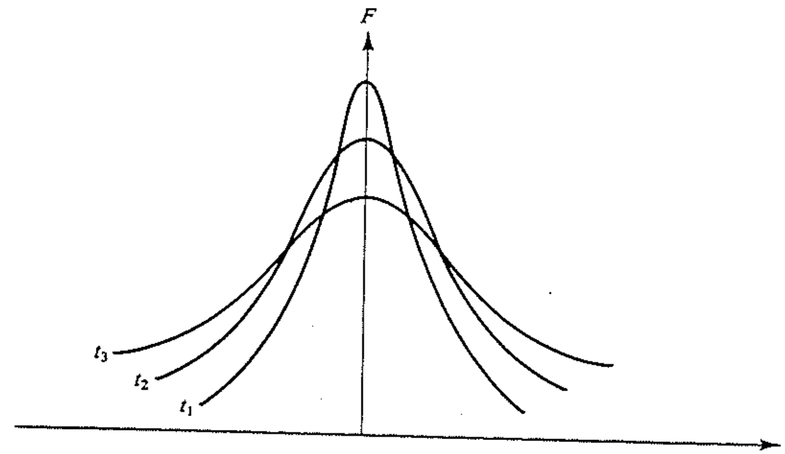

FIGURE 1.3. Fundamental solution for the temperature at three different times

on t? It follows from (1.2.20) and symmetry that

$$d = \frac{2}{2\sqrt{\pi t}} \int_0^{x_d} e^{-\sigma^2/4t} d\sigma,$$

or, changing variables, that

$$d = \frac{2}{\sqrt{\pi}} \int_0^{x_d/2\sqrt{t}} e^{-\eta^2} d\eta \equiv \operatorname{erf}\left(\frac{x_d}{2\sqrt{t}}\right), \tag{1.2.23}$$

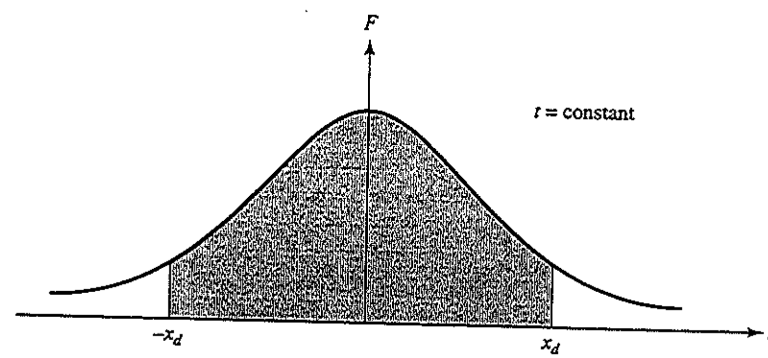

FIGURE 1.4. Interval $(-x_d, x_d)$ containing the fraction d of the total heat

where the error function erf is defined in (A.2.76). Since the left-hand side of (1.2.23) is a constant, we conclude that $x_d/2\sqrt{t}$ remains constant as t increases. Therefore, $x_d \sim \sqrt{t}$, and we say that heat due to a point source at $x = 0$, $t = 0$ *diffuses* according to $|x| \sim \sqrt{t}$.

Problems

1.2.1. Consider the diffusion equation with variable coefficient

$$2xu_t - u_{xx} = 0, \quad 0 \le x < \infty, \quad t \ge 0, \tag{1.2.24}$$

with boundary conditions

$$u(0,t) = C_1 = \text{constant} \quad \text{if } t > 0, \tag{1.2.25a}$$

$$u(\infty,t) = C_2 = \text{constant} \quad \text{if } t > 0, \tag{1.2.25b}$$

and initial condition

$$u(x,0) = C_3 = \text{constant}. \tag{1.2.26}$$

a. What is the most general choice for the constants C_1, C_2, and C_3 for which the solution of the above initial- and boundary-value problem can be obtained in similarity form?

b. For the choice of constants obtained in part (a), calculate the solution and evaluate all integration constants explicitly.

1.2.2. Use similarity to reduce the following initial- and boundary-value problem for a nonlinear diffusion equation to an ordinary differential equation and corresponding boundary conditions:

$$u_{xx} - uu_t = 0, \quad 0 \le x, \quad 0 \le t, \tag{1.2.27}$$

$$u(0,t) = 0, \tag{1.2.28b}$$

$$u(\infty,t) = 1, \tag{1.2.28b}$$

$$u(x,0) = 1. \tag{1.2.29}$$

Discuss the behavior of the solution.

1.2.3 A semi-infinite bar ($x \ge 0$) insulated everywhere except at $x = 0$ loses heat to the adjacent medium ($x < 0$) by blackbody radiation according to the boundary condition

$$\theta^4(0,t) - \theta_0^4 = \alpha\theta_x(0,t), \quad t > 0, \tag{1.2.30}$$

where α is a constant (equal to the conductivity divided by the product of the emissivity and the Stefan-Boltzmann constant), θ_0 is the constant temperature of the medium, and $\theta(x,t)$ is the temperature at the point x and time t in the bar. Equation (1.1.9) with $\sigma \equiv 0$ governs the temperature distribution in $x \ge 0$, and we assume that the initial temperature is given

in the form

$$\theta(x,0) = \theta_1 f\left(\frac{x}{L_0}\right), \tag{1.2.31}$$

where θ_1 is a characteristic temperature and L_0 is a characteristic length. The boundary condition at $x = \infty$ is

$$\theta(\infty,t) = \theta_1 f(\infty) < \infty. \tag{1.2.32}$$

a. Introduce appropriate dimensionless variables u, x^*, t^* to reduce (1.2.30)–(1.2.32) to the form

$$\frac{\partial u}{\partial t^*} - \frac{\partial^2 u}{\partial x^{*2}} = 0, \tag{1.2.33a}$$

$$u(x^*,0) = \frac{1}{\epsilon} f(x^*), \tag{1.2.33b}$$

$$u^4(0,t^*) - 1 = \lambda \frac{\partial u}{\partial x^*}(0,t^*), \quad t^* > 0, \tag{1.2.33c}$$

$$u(\infty,t^*) = \frac{1}{\epsilon} f(\infty), \tag{1.2.33d}$$

where ϵ and λ are dimensionless constants.

b. What does the limiting case

$$\lambda \gg 1, \quad \epsilon \ll 1, \quad \lambda\epsilon^3 = \text{constant} = \bar{\lambda} = O(1), \tag{1.2.34}$$

describe physically? Since u is initially large, it is appropriate to consider the rescaled dependent variable $\bar{u} = u/\epsilon$, where $\bar{u}$ is $O(1)$. Thus, to leading order, $\bar{u}$ satisfies

$$\frac{\partial \bar{u}}{\partial t^*} - \frac{\partial^2 \bar{u}}{\partial x^{*2}} = 0, \tag{1.2.34a}$$

$$\bar{u}(x^*,0) = f(x^*), \tag{1.2.34b}$$

$$\bar{u}^4(0,t^*) = \bar{\lambda}\frac{\partial \bar{u}}{\partial x^*}(0,t^*) + O(\epsilon^4), \quad t^* > 0, \tag{1.2.34c}$$

$$\bar{u}(\infty,t^*) = f(\infty). \tag{1.2.34d}$$

c. For what $f(x^*)$ (possibly singular) can (1.2.34) be solved by similarity? For this choice of f derive, but do not solve, the ordinary differential equation and boundary conditions governing the solution.

1.3 Initial-Value Problem in the Infinite Domain; Superposition

The general initial-value problem for the inhomogeneous diffusion equation in the infinite interval is

$$u_t - u_{xx} = p(x,t), \quad -\infty < x < \infty, \quad t \ge 0, \tag{1.3.1a}$$

$$u(x, 0^+) = f(x), \tag{1.3.1b}$$

where p and f are arbitrarily prescribed functions with $p \equiv 0$ if $t < 0$. For heat conduction, p represents a dimensionless heat-source distribution, and f an initial temperature distribution.

Because of linearity, the solution of (1.3.1) can be expressed as the sum of the following two problems:

$$u_t - u_{xx} = p(x, t), \quad -\infty < x < \infty, \quad t \geq 0, \tag{1.3.2a}$$

$$u(x, 0^-) = 0, \tag{1.3.2b}$$

$$u_t - u_{xx} = 0, \quad -\infty < x < \infty, \quad t \geq 0, \tag{1.3.3a}$$

$$u(x, 0^+) = f(x). \tag{1.3.3b}$$

We now show that knowing the fundamental solution $F(x - \xi, t - \tau)$ allows us to write the solution of the first problem immediately in terms of a "superposition integral." The derivation of this superposition integral is a straightforward generalization of the single-variable case discussed in Appendix 1 (see (A.1.23)–(A.1.28)). We consider the solution of (1.3.2) arising from the contribution of p coming from a small neighborhood of the fixed point $x = \xi, t = \tau$, with p set equal to zero everywhere outside this neighborhood. Let $R(\xi, \tau)$ denote the small neighborhood $\xi - \Delta\xi/2 \leq x \leq \xi + \Delta\xi/2$, $\tau - \Delta\tau/2 \leq t \leq \tau + \Delta\tau/2$, over which we may regard the value of p as the constant $p(\xi, \tau)$.

If $\tilde{p}$ denotes the incremental contribution to p from R, we have the following expression defining $\tilde{p}$:

$$\tilde{p} \equiv p(\xi, \tau)\left[H\left(t - \tau + \frac{\Delta\tau}{2}\right) - H\left(t - \tau - \frac{\Delta\tau}{2}\right)\right]\left[H\left(x - \xi + \frac{\Delta\xi}{2}\right) - H\left(x - \xi - \frac{\Delta\xi}{2}\right)\right], \tag{1.3.4}$$

where H is the Heaviside function, and the bracketed expressions ensure that the left-hand side vanishes outside R and equals $\tilde{p}$ in R. We now multiply and divide this expression for $\tilde{p}$ by $\Delta\tau\Delta\xi$ and observe that since $dH/ds = \delta(s)$, the first bracketed expression divided by $\Delta\tau$ represents $\delta(t - \tau)$, whereas the second bracketed expression divided by $\Delta\xi$ represents $\delta(x - \xi)$. Therefore, in the "limit" as $\Delta\tau \to 0$, $\Delta\xi \to 0$, we have

$$\tilde{p} = p(\xi, \tau)\delta(t - \tau)\delta(x - \xi)d\tau\, d\xi. \tag{1.3.5}$$

Since the solution of the diffusion equation with right-hand side $\delta(t - \tau)\delta(x - \xi)$ is the fundamental solution $F(x - \xi, t - \tau)$ defined in (1.2.21), linearity implies that the solution due to the right-hand side $\tilde{p}$ is just

$$\tilde{u} = p(\xi, \tau)F(x - \xi, t - \tau)d\tau\, d\xi. \tag{1.3.6}$$

Linearity also implies that we may superpose the $\tilde{u}$ contributions arising from each of the infinitesimal domains R that cover the half-space $-\infty < \xi < \infty$, $0 \leq \tau < t$, and this leads to the desired superposition integral

$$u(x, t) = \int_{\xi=-\infty}^{\infty}\int_{\tau=0^-}^{t} F(x - \xi, t - \tau)p(\xi, \tau)d\tau\, d\xi$$

$$= \int_{\xi=-\infty}^{\infty}\int_{\tau=0^-}^{t} \frac{p(\xi, \tau)}{2\sqrt{\pi(t - \tau)}}e^{-(x-\xi)^2/4(t-\tau)}d\tau\, d\xi. \tag{1.3.7}$$

To confirm this formal derivation, it is easy to verify explicitly that (1.3.7) solves (1.3.2); this is left as an exercise (Problem 1.3.1).

To solve (1.3.3), we note that it is equivalent to

$$u_t - u_{xx} = \delta(t)f(x), \quad -\infty < x < \infty, \quad t \geq 0, \tag{1.3.8a}$$

$$u(x, 0^-) = 0, \tag{1.3.8b}$$

as can be verified by noting that integrating the inhomogeneous diffusion equation (1.3.8a) with respect to t from $t = 0^-$ to $t = 0^+$ gives $u(x, 0^+) = f(x)$. Since the right-hand side of (1.3.8a) vanishes when $t > 0$, (1.3.3) and (1.3.8) are equivalent. To solve (1.3.8), we set $p(\xi, \tau)$ in (1.3.7) equal to $\delta(\tau)f(\xi)$ and obtain

$$u(x, t) = \frac{1}{2\sqrt{\pi t}}\int_{\xi=-\infty}^{\infty} f(\xi)e^{-(x-\xi)^2/4t}d\xi. \tag{1.3.9}$$

This result can also be derived using transforms (see (A.2.32)–(A.2.36) for a derivation using Fourier transforms). Therefore, the solution of (1.3.1) is the sum of the solutions (1.3.7) and (1.3.9).

Note that

$$u(x, 0^+) = \lim_{t\to 0^+}\int_{-\infty}^{\infty} f(\xi)\frac{e^{-(x-\xi)^2/4t}}{2\sqrt{\pi t}}d\xi, \tag{1.3.10}$$

and according to (1.2.22), this is just

$$u(x, 0^+) = \int_{-\infty}^{\infty} f(\xi)\delta(x - \xi)d\xi = f(x), \tag{1.3.11}$$

which is the correct initial condition.

We can also verify that the initial condition is satisfied by the following alternative approach that does not involve use of the delta function. We write (1.3.9), as the sum of three integrals over the intervals $(-\infty, x - \epsilon)$, $(x - \epsilon, x + \epsilon)$, and $(x + \epsilon, \infty)$, where ϵ is an arbitrarily small, fixed positive number. As $t \to 0^+$, the integrals tend to zero except over the interval $(x - \epsilon, x + \epsilon)$. Thus,

$$u(x, 0^+) = \lim_{t\to 0^+}\int_{x-\epsilon}^{x+\epsilon} f(\xi)\frac{e^{-(x-\xi)^2/4t}}{2\sqrt{\pi t}}d\xi. \tag{1.3.12}$$

Changing the variable of integration from ξ to $\sigma = (x - \xi)/2t^{1/2}$ gives

$$u(x, 0^+) = \lim_{t\to 0^+}\frac{1}{\sqrt{\pi}}\int_{-\epsilon/\sqrt{4t}}^{\epsilon/\sqrt{4t}} f(x + \sigma\sqrt{4t})e^{-\sigma^2}d\sigma$$

$$= \frac{f(x)}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\sigma^2}d\sigma = f(x). \tag{1.3.13}$$

Problems

1.3.1. Verify by direct substitution that the sum of the expressions given by (1.3.7) and (1.3.9) solves the initial-value problem (1.3.1).

1.3.2a Specialize (1.3.9) to the case where

$$f(x) = \begin{cases} 1/2\epsilon, & -\epsilon < x < \epsilon, \\ 0, & |x| > \epsilon, \end{cases} \tag{1.3.14}$$

and show that the solution reduces to

$$u(x,t) = \frac{1}{4\epsilon}\left[\operatorname{erf}\left(\frac{x+\epsilon}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-\epsilon}{2\sqrt{t}}\right)\right], \tag{1.3.15}$$

where the error function erf is defined in (A.2.76).

b. Show that as $\epsilon \to 0$ the result in (1.3.15) tends to the fundamental solution (1.2.20), as expected, since (1.3.14) is a representation of the delta function (see (A.1.3)), and the solution (1.3.9) with $f(x) = \delta(x)$ is just (1.2.20).

1.3.3. Specialize (1.3.7) to the case where $p(x,t)$ is a uniformly moving source

$$p(x,t) = \delta(x - vt), \quad v = \text{constant}. \tag{1.3.16}$$

1.4 Problems in the Semi-infinite Domain; Green's Functions

In studying the diffusion equation over the semi-infinite interval with a prescribed boundary condition at $x = 0$, it is useful first to consider the solution that results from a unit source somewhere in the domain and subject to a homogeneous (zero) boundary condition at the origin. This solution will be denoted by Green's function of the first kind, G_1, or second kind, G_2, depending on whether the boundary condition at $x = 0$ is $u = 0$ or $u_x = 0$.

1.4.1 Green's Function of the First Kind

Consider first the case where $u = 0$ at the origin; that is, we seek the solution for

$$u_t - u_{xx} = \delta(t)\delta(x - \xi) \tag{1.4.1a}$$

on $0 \le x < \infty$, with ξ equal to a positive constant, and impose the boundary condition

$$u(0,t) = 0, \quad t > 0, \tag{1.4.1b}$$

and initial condition

$$u(x, 0^-) = 0. \tag{1.4.1c}$$

(Unless stated otherwise, we shall take the boundary condition for u at $x = \infty$ to be the same as the limit as $x \to \infty$ of the initial value. Thus, in the present case, we have $u(\infty, t) = u(\infty, 0) = 0$.)

Thus, we have introduced a concentrated unit source of heat at $x = \xi$ and $t = 0$. (Note that we can derive the solution for the case where (1.4.1) involves $\delta(t - \tau)$ by replacing t everywhere in the solution by $t - \tau$.) The rod is initially at zero temperature, and its left end is maintained at zero temperature for all time, for example, by attaching this end to an infinite solid of zero temperature.

The only difference between this problem and the fundamental solution is the fact that we require u to vanish at $x = 0$ and $x \to \infty$ instead of $x \to \pm\infty$. Thus, Green's function is the response to a source with a homogeneous boundary condition imposed at a finite point.

An intuitively appealing procedure invokes symmetry relative to the origin to construct the solution once the fundamental solution is known. (This is often called the *method of images*.)

Consider the temperature that results in the *infinite* domain if we turn on a positive source of unit strength at $x = \xi$ and $t = 0$, and *simultaneously* turn on a negative source of unit strength at $x = -\xi$, the image point.

At any time $t > 0$, the temperature in the rod will be the sum of the two temperatures $F(x - \xi, t)$ and $-F(x + \xi, t)$, corresponding to the positive and negative sources, respectively. These individual temperature profiles at some $t > 0$ are sketched in Figure 1.5. In particular, the combined temperature will always vanish at $x = 0$ for $t > 0$, by symmetry. Moreover, since the image source is located at $x = -\xi$, outside the domain of interest, the combined temperature satisfies (1.4.1a). Therefore, the solution of (1.4.1) is Green's function:

$$G_1(x, \xi, t) \equiv F(x - \xi, t) - F(x + \xi, t), \tag{1.4.2}$$

where F is defined by (1.2.20).

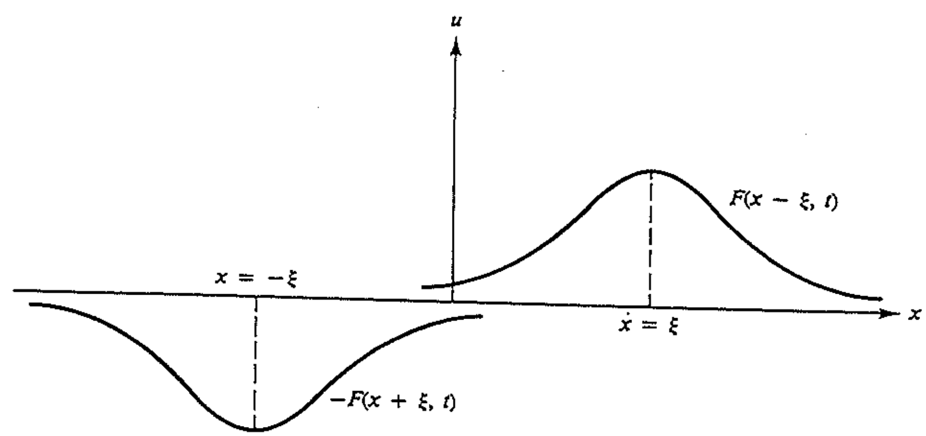

FIGURE 1.5. Temperature due to a unit positive source at $x = \xi$ and a unit negative source at $x = -\xi$

More generally, the solution of

$$u_t - u_{xx} = \delta(x-\xi)\delta(t-\tau), \quad \xi > 0, \quad \tau > 0, \qquad (1.4.3)$$

with initial condition $u(x, \tau^-) = 0$ and boundary condition $u(0, t) = 0$ for $t > \tau$ and x on the semi-infinite interval $0 \le x < \infty$ is Green's function of the first kind for the semi-infinite domain and has the form

$$G_1(x, \xi, t-\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}}[e^{-(x-\xi)^2/4(t-\tau)} - e^{-(x+\xi)^2/4(t-\tau)}]. \qquad (1.4.4)$$

1.4.2 Homogeneous Boundary-Value Problems

Consider the following *inhomogeneous* diffusion equation with zero initial condition and *homogeneous* boundary condition:

$$u_t - u_{xx} = p(x, t), \quad 0 \le x, \quad 0 \le t, \qquad (1.4.5a)$$

$$u(x, 0^-) = 0, \qquad (1.4.5b)$$

$$u(0, t) = 0, \quad t > 0. \qquad (1.4.5c)$$

The superposition idea leading to (1.3.7) also applies for this case, and we have

$$u(x, t) = \int_{0^-}^{t} d\tau \int_0^\infty p(\xi, \tau) G_1(x, \xi, t-\tau) d\xi. \qquad (1.4.6)$$

It is important to bear in mind that *Green's function and the desired solution of (1.4.5) must both satisfy a zero boundary condition at the origin* in order for the superposition idea and the result (1.4.6) to make sense. For example, if $G_1(0, \xi, t-\tau) \neq 0$, then (1.4.6) does not satisfy (1.4.5c). Conversely, if we wish to solve the problem (1.4.5) with the right-hand side of (1.4.5c) replaced by some prescribed function $g(t)$, the representation (1.4.6) fails, since it automatically has $u(0, t) = 0$. We shall see in Section 1.4.3 that this case is easily handled once the problem is transformed to one with a zero boundary condition at the origin.

Consider now the case where the initial condition (1.4.5b) is prescribed arbitrarily. Since the homogeneous problem

$$u_t - u_{xx} = 0, \quad 0 \le x, \quad 0 \le t, \qquad (1.4.7a)$$

with nonzero initial condition

$$u(x, 0^+) = f(x) \qquad (1.4.7b)$$

and homogeneous boundary condition

$$u(0, t) = 0, \quad t > 0, \qquad (1.4.7c)$$

is equivalent to

$$u_t - u_{xx} = \delta(t) f(x) \qquad (1.4.8)$$

with $u(x, 0^-) = 0$ and $u(0, t) = 0$, we can express the solution of (1.4.7) using the result (1.4.6) with $p = \delta(\tau) f(\xi)$; that is,

$$u(x, t) = \int_0^\infty \int_{0^-}^{t} \delta(\tau) f(\xi) G_1(x, \xi, t-\tau) d\tau d\xi = \int_0^\infty f(\xi) G_1(x, \xi, t) d\xi. \qquad (1.4.9)$$

For the special case where $f(\xi) = c$, a constant, (1.4.9) gives

$$u(x, t) = \frac{c}{2\sqrt{\pi t}} \left[\int_0^\infty e^{-(x-\xi)^2/4t} d\xi - \int_0^\infty e^{-(x+\xi)^2/4t} d\xi \right]. \qquad (1.4.10)$$

Changing the variable of integration from ξ to $\eta = (x-\xi)/2t^{1/2}$ in the first integral and to $\eta = (x+\xi)/2t^{1/2}$ in the second integral results in

$$u(x, t) = \frac{c}{\sqrt{\pi}} \left\{ -\int_{x/2\sqrt{t}}^{0} e^{-\eta^2} d\eta - \int_0^{-\infty} e^{-\eta^2} d\eta - \int_{x/2\sqrt{t}}^{\infty} e^{-\eta^2} d\eta \right\}. \qquad (1.4.11a)$$

It is important to note that because $x - \xi$ vanishes for $\xi = x$, which is a point in $(0, \infty)$, the first integral in (1.4.11a) must be decomposed into two parts. Simplifying this expression gives

$$u(x, t) = \frac{2c}{\sqrt{\pi}} \int_0^{x/2\sqrt{t}} e^{-\eta^2} d\eta = c \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right), \qquad (1.4.11b)$$

where the error function erf is defined in (A.2.76).

The qualitative behavior of the solution (1.4.11) has u rising rapidly from its zero boundary value to the asymptotic value $u = c$. Temperature profiles at various times are sketched in Figure 1.6.

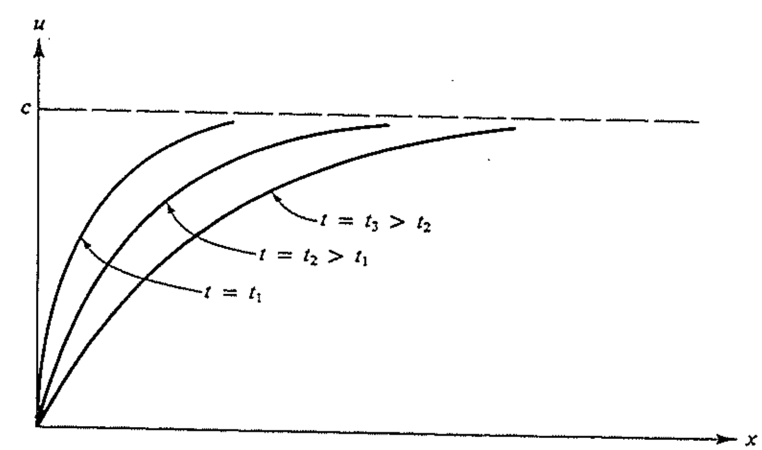

FIGURE 1.6. Temperature profiles at various values of t

Notice that $\lim_{\substack{t\to 0^+ \\ x>0}} u(x,t) = c$, in agreement with (1.4.7b), and that $\lim_{\substack{x\to 0^+ \\ t>0}} u(x,t) = 0$, in agreement with (1.4.7c). In particular, $u(0^+, 0^+)$ is undefined, as is to be expected from (1.4.7b) and (1.4.7c).

1.4.3 Inhomogeneous Boundary Condition $u(0,t) = g(t)$

As pointed out in Section 1.4.2, the crucial requirement for applying superposition is that the boundary condition at $x = 0$ be homogeneous. Does this mean that we cannot use Green's functions to solve an inhomogeneous boundary-value problem? We shall show next that if it is possible to transform the problem to one with a homogeneous boundary condition at $x = 0$ (as is often the case), a solution derived by superposition of Green's functions can still be used.

Consider the inhomogeneous boundary-value problem

$$u_t - u_{xx} = 0, \quad 0 \le x < \infty, \quad 0 \le t < \infty, \tag{1.4.12a}$$

with zero initial condition

$$u(x, 0^+) = 0, \tag{1.4.12b}$$

and a prescribed boundary condition at $x = 0$:

$$u(0,t) = g(t), \quad t > 0. \tag{1.4.12c}$$

Again, in view of (1.4.12b), it is understood that $u(\infty, t) = 0$.

The idea is to transform $u(x,t)$ to a new dependent variable $w(x,t)$, which obeys a homogeneous boundary condition at the origin. Clearly, the simple *homogenizing transformation*

$$w(x,t) \equiv u(x,t) - g(t) \tag{1.4.13}$$

works, since w obeys the inhomogeneous diffusion equation

$$w_t - w_{xx} = -\dot{g}(t), \quad t > 0, \tag{1.4.14}$$

with constant initial condition

$$w(x, 0^+) = -g(0^+), \tag{1.4.15a}$$

and zero boundary condition

$$w(0,t) = 0, \quad t > 0. \tag{1.4.15b}$$

Note that $w(\infty, t) = -g(t)$ if $t > 0$, but this does not preclude superposition. A problem equivalent to (1.4.14)–(1.4.15) is

$$w_t - w_{xx} = -\dot{g}(t) - g(0^+)\delta(t), \tag{1.4.16a}$$

$$w(x, 0^-) = 0, \tag{1.4.16b}$$

$$w(0,t) = 0, \quad t > 0, \tag{1.4.16c}$$

and the system (1.4.16) is a special case of (1.4.5), with $p(x,t) = -\dot{g}(t) - g(0^+)\delta(t)$. Writing out the solution (1.4.6) for this case gives

$$\begin{aligned} w(x,t) = &\int_{0^+}^{t}\int_0^\infty \frac{-\dot{g}(\tau)}{2\sqrt{\pi(t-\tau)}}[e^{-(x-\xi)^2/4(t-\tau)} - e^{-(x+\xi)^2/4(t-\tau)}]d\xi\, d\tau \\ &- \int_0^\infty \frac{g(0^+)}{2\sqrt{\pi t}}[e^{-(x-\xi)^2/4t} - e^{-(x+\xi)^2/4t}]d\xi. \end{aligned} \tag{1.4.17}$$

The solution (1.4.17) involves the two integrals

$$I = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-(x-\xi)^2/4(t-\tau)}}{2\sqrt{t-\tau}}d\xi \tag{1.4.18a}$$

and

$$K = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-(x+\xi)^2/4(t-\tau)}}{2\sqrt{t-\tau}}d\xi. \tag{1.4.18b}$$

In preparation for evaluating I, we set the exponent in the integrand equal to $-\eta^2$, where η is a new variable of integration. Again, we must be careful to take into account the fact that this exponent vanishes at the point $\xi = x > 0$, which is inside the interval of integration. Thus, we first split (1.4.18a) into two integrals over $0 \le \xi \le x$ and $x \le \xi < \infty$; then we change variables $\xi \to \eta$ by setting $(x-\xi)/2\sqrt{t-\tau} = \eta$, $d\xi = -2\sqrt{t-\tau}\,d\eta$ to obtain

$$\begin{aligned} I &= \frac{1}{\sqrt{\pi}}\left[\int_{x/2\sqrt{t-\tau}}^0 e^{-\eta^2}(-d\eta) + \int_0^{-\infty} e^{-\eta^2}(-d\eta)\right] \\ &= \frac{1}{\sqrt{\pi}}\left[\int_0^{x/2\sqrt{t-\tau}} e^{-\eta^2}d\eta + \int_0^\infty e^{-\eta^2}d\eta\right]. \end{aligned}$$

It then follows from the definition (A.2.76) of the error function that

$$I = \frac{1}{2}\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) + \frac{1}{2}. \tag{1.4.18c}$$

Since $(x+\xi)$ does not vanish for $x > 0$ if $0 \le \xi < \infty$, we evaluate K directly by setting $(x+\xi)/2\sqrt{t-\tau} = \eta$ to obtain

$$K = \frac{1}{\sqrt{\pi}}\left[\int_{x/2\sqrt{t-\tau}}^\infty e^{-\eta^2}d\eta\right] = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right), \tag{1.4.18d}$$

where erfc denotes the complementary error function, $\operatorname{erfc}(y) = 1 - \operatorname{erf}(y)$. See (A.2.77).

Thus, I may also be written as

$$I = 1 - \frac{1}{2}\operatorname{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right), \tag{1.4.19}$$

and (1.4.17) becomes

$$w(x,t) = \int_{0^+}^{t} \dot{g}(\tau)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right)d\tau + g(0^+)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right) - g(t). \tag{1.4.20}$$

Therefore, $u(x,t) = w(x,t) + g(t)$ is given by

$$u(x,t) = \int_{0^+}^{t} \dot{g}(\tau)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right)d\tau + g(0^+)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right). \tag{1.4.21}$$

Note that $w(\infty,t) = -g(t)$ as required. For the special case $g(t) = d =$ constant, $\dot{g} = 0$, and we have $u(x,t) = d\,\mathrm{erfc}(x/2\sqrt{t})$. Here, again, as for (1.4.11), $u(0^+,0^+)$ is undefined. However, $\lim_{\substack{t\to 0^+\\ x>0}} u(x,t) = 0$, in agreement with (1.4.12b), and $\lim_{\substack{x\to 0^+\\ t>0}} u(x,t) = d$, in agreement with (1.4.12c).

Integrating the first term by parts in (1.4.21) gives the alternative form

$$u(x,t) = \frac{x}{2\sqrt{\pi}}\int_0^t \frac{g(\tau)e^{-x^2/4(t-\tau)}}{[t-\tau]^{3/2}}d\tau = \frac{x}{2\sqrt{\pi}}\int_0^t \frac{g(t-\tau)e^{-x^2/4\tau}}{\tau^{3/2}}d\tau. \tag{1.4.22}$$

The solution (1.4.22) is derived in Appendix A.2 using Laplace transforms. See (A.2.73). In Problem 1.4.4a this result is obtained as the solution of a related integral equation. Problem 1.4.6 explores the application of the preceding ideas to the case of discontinuous material properties. Problem 1.4.7 concerns the effect of moving boundaries.

Next, we consider problems on the semi-infinite domain subject to the homogeneous boundary condition $u_x = 0$ at $x = 0$ and see how Green's function may also be used to solve the problem where u_x is specified at $x = 0$.

1.4.4. Green's Function of the Second Kind

We can also use a symmetry argument to solve

$$u_t - u_{xx} = \delta(x-\xi)\delta(t) \tag{1.4.23a}$$

on $0 \le x < \infty$, with $\xi > 0$ subject to the boundary condition

$$u_x(0,t) = 0, \quad t > 0, \tag{1.4.23b}$$

and initial condition

$$u(x,0^-) = 0. \tag{1.4.23c}$$

Here again, we assume that as $x \to \infty$, u remains equal to the value it has at infinity initially.

We might interpret the solution of (1.4.23) as the temperature in a semi-infinite rod in response to a unit source of heat at $x = \xi$, $t = 0$ for the case where the rod is insulated (that is, there is no heat flow) at the left end.

In order to ensure that condition (1.4.23b) holds for all $t > 0$ at the origin, we need to introduce an *image*, or *reflected*, source of unit *positive* strength at the image point $x = -\xi$. The situation corresponding to Figure 1.5 now has the two bell-shaped profiles above the x-axis and centered at the points $x = \pm\xi$. Therefore, the slope of the combined profile vanishes at $x = 0$, since the contributions to u_x from the source at $x = \xi$ and $x = -\xi$ cancel out exactly for all $t > 0$.

Thus, the solution of (1.4.23) is

$$G_2(x,\xi,t) \equiv F(x-\xi,t) + F(x+\xi,t), \tag{1.4.24}$$

where F is the fundamental solution defined by (1.2.20).

More generally, if the source is turned on at $t = \tau > 0$, we have

$$G_2(x,\xi,t-\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}}[e^{-(x-\xi)^2/4(t-\tau)} + e^{-(x+\xi)^2/4(t-\tau)}]. \tag{1.4.25}$$

1.4.5 Homogeneous Boundary-Value Problems

As in Section 1.4.2 we can use superposition to express the solution of

$$u_t - u_{xx} = p(x,t), \quad 0 \le x, \ 0 \le t, \tag{1.4.26a}$$

$$u(x,0^-) = 0, \tag{1.4.26b}$$

$$u_x(0,t) = 0, \quad t > 0, \tag{1.4.26c}$$

in the form

$$u(x,t) = \int_0^t d\tau \int_0^\infty [p(\xi,\tau)G_2(x,\xi,t-\tau)]d\xi. \tag{1.4.27}$$

Also, we can accommodate a nonzero initial condition

$$u(x,0^+) = f(x)$$

by adding to (1.4.27) the contribution

$$u(x,t) = \int_0^\infty f(\xi)G_2(x,\xi,t)d\xi. \tag{1.4.28}$$

For the case $f(\xi) = c =$ constant, it is easily seen by changing the sign of the second term in (1.4.10) that (1.4.28) reduces to $u = c$, as expected.

1.4.6 Inhomogeneous Boundary Condition, $u_x(0,t) = h(t)$

To solve the problem

$$u_t - u_{xx} = 0, \quad 0 \le x < \infty, \quad 0 \le t < \infty, \tag{1.4.29a}$$

$$u(x,0^+) = 0, \tag{1.4.29b}$$

$$u_x(0,t) = h(t), \quad t > 0, \tag{1.4.29c}$$

we introduce the homogenizing transformation

$$w(x,t) \equiv u(x,t) - xh(t). \tag{1.4.30}$$

It then follows that if u solves (1.4.29), w solves

$$w_t - w_{xx} = -x\dot{h}(t), \tag{1.4.31a}$$

$$w(x, 0^+) = -xh(0^+), \tag{1.4.31b}$$

$$w_x(0, t) = 0. \tag{1.4.31c}$$

Using the results in (1.4.27) and (1.4.28), we have

$$u(x, t) - xh(t) = -\int_0^t d\tau \int_0^\infty \xi \dot{h}(\tau) G_2(x, \xi, t - \tau) d\xi$$
$$- h(0^+) \int_0^\infty \xi G_2(x, \xi, t) d\xi. \tag{1.4.32a}$$

This can be simplified to the form

$$u(x, t) = -\frac{1}{\sqrt{\pi}} \int_0^t h(\tau)(t - \tau)^{-1/2} e^{-x^2/4(t-\tau)} d\tau. \tag{1.4.32b}$$

In Problem 1.4.8 you are asked to derive this result and to reconcile it with the result obtained by Laplace transforms.

1.4.7 The General Linear Boundary-Value Problem

The general linear boundary-value problem over the semi-infinite domain is

$$u_t - u_{xx} = p(x, t), \tag{1.4.33a}$$

$$u(x, 0^+) = 0, \tag{1.4.33b}$$

$$a(t)u(0, t) + b(t)u_x(0, t) = c(t), \quad t > 0, \tag{1.4.33c}$$

as we have the most general linear boundary condition (1.4.33c) at the left end with arbitrarily prescribed nonvanishing functions a, b, and c. In our previous discussion, we have solved the two special cases $a = 0$ and $b = 0$. There is no loss of generality in setting $u(x, 0^+) = 0$ in (1.4.33b), since for a general initial condition $u(x, 0^+) = f(x)$, we can transform the problem to the form (1.4.33) by considering $u - f$ as a new dependent variable.

A Green's function approach is not feasible if a, b, and c are all nonzero, and we study two approaches next for solving (1.4.33).

(i) $au(0, t) + bu_x(0, t) = c(t)$, *a and b constant*

If a and b are constant, (1.4.33c) may be interpreted as Newton's law of cooling for a semi-infinite heat conductor with its left end ($x = 0$) in contact with a heat reservoir with prescribed time-dependent temperature. We write (1.4.33c) in the form

$$bu_x(0, t) = a[u_R(t) - u(0, t)], \tag{1.4.34}$$

and regard $-b$ as the thermal conductivity, $a > 0$ as the heat transfer coefficient, and $u_R(t) = c(t)/a$ as the reservoir temperature. Thus, for example, if $u_R(t) >$

$u(0, t)$, we expect heat to flow from the reservoir into the conductor, making the left end $x = 0$ hotter than the interior, i.e., $u_x(0, t) < 0$. This follows from (1.4.34), since $b < 0$ in this interpretation.

One approach for solving (1.4.33) is to introduce a new dependent variable $v(x, t)$ defined by

$$v(x, t) \equiv au(x, t) + bu_x(x, t). \tag{1.4.35}$$

If we compute $v_t - v_{xx}$ using (1.4.35) we obtain

$$v_t - v_{xx} = a(u_t - u_{xx}) + b(u_t - u_{xx})_x.$$

Thus, if u satisfies (1.4.33a), v satisfies

$$v_t - v_{xx} = ap(x, t) + bp_x(x, t) \equiv q(x, t), \tag{1.4.36a}$$

the same diffusion equation with a different, but known, right-hand side. Note that if a and b depend on t, this approach *does not* lead to the same diffusion equation; we pick up additional terms involving time-dependent coefficients.

The initial and boundary conditions for v are obtained in the form

$$v(x, 0) = 0, \tag{1.4.36b}$$

$$v(0, t) = c(t). \tag{1.4.36c}$$

Therefore, using (1.4.6) and (1.4.9), we have

$$v(x, t) = \frac{x}{2\sqrt{\pi}} \int_0^t \tau^{-3/2} c(t - \tau) e^{-x^2/4\tau} d\tau$$
$$+ \int_0^t d\tau \int_0^\infty q(\xi, \tau) G_1(x, \xi, t - \tau) d\xi, \tag{1.4.37}$$

where G_1 is defined in (1.4.4).

Knowing $v(x, t)$, we compute $u(x, t)$ by solving the linear inhomogeneous ordinary differential equation (1.4.35). This gives

$$u(x, t) = \phi(t) e^{-ax/b} + \frac{e^{-ax/b}}{b} \int_0^x v(\xi, t) e^{a\xi/b} d\xi, \tag{1.4.38}$$

where $\phi(t)$ is as yet unspecified. The initial condition $u(x, 0) = 0$ and the fact that $v(x, 0) = 0$ imply that $\phi(0) = 0$. It is easy to verify by direct substitution that (1.4.38) satisfies the boundary condition (1.4.33c) identically. To determine $\phi(t)$ we substitute (1.4.38) into the governing equation (1.4.33a). We have

$$u_t = \dot{\phi}(t) e^{-ax/b} + \frac{1}{b} e^{-ax/b} \int_0^x v_t(\xi, t) e^{a\xi/b} d\xi, \tag{1.4.39a}$$

$$u_x = -\frac{a}{b} \phi(t) e^{-ax/b} - \frac{a}{b^2} e^{-ax/b} \int_0^x v(\xi, t) e^{a\xi/b} d\xi + \frac{1}{b} v(x, t), \tag{1.4.39b}$$

$$u_{xx} = \frac{a^2}{b^2} \phi(t) e^{-ax/b} + \frac{a^2}{b^3} e^{-ax/b} \int_0^x v(\xi, t) e^{a\xi/b} d\xi - \frac{a}{b^2} v(x, t)$$
$$+ \frac{1}{b} v_x(x, t). \tag{1.4.39c}$$

The integral in the expression defining u_{xx} can be developed by integration by parts twice to give

$$u_{xx} = \frac{a^2}{b^2}\phi(t)e^{-ax/b} - \frac{a}{b^2}v(0,t)e^{-ax/b} + \frac{1}{b}v_x(0,t)e^{-ax/b} + \frac{1}{b}e^{-ax/b}\int_0^x v_{xx}(\xi,t)e^{a\xi/b}d\xi. \tag{1.4.39d}$$

Substituting (1.4.39a) for u_t and (1.4.39d) for u_{xx} into (1.4.33a) and using the boundary condition (1.4.33c) gives

$$p(x,t) = \left[\dot{\phi}(t) - \frac{a^2}{b^2}\phi(t)\right]e^{-ax/b} + \frac{1}{b}e^{-ax/b}\int_0^x [ap(\xi,t) + bp_x(\xi,t)]e^{a\xi/b}d\xi + \left[\frac{a}{b^2}c(t) - \frac{1}{b}v_x(0,t)\right]e^{-ax/b}. \tag{1.4.40}$$

Now, when we integrate by parts the integral of $pe^{a\xi/b}$, the integrals involving p_x cancel. We also pick up a $p(x,t)$ on the right-hand side of (1.4.40) that cancels the $p(x,t)$ on the left-hand side. Finally, we multiply though by $e^{ax/b}$ to obtain the following first-order linear inhomogeneous ordinary differential equation governing $\phi(t)$:

$$\dot{\phi}(t) - \frac{a^2}{b^2}\phi(t) = -\frac{a}{b^2}c(t) + p(0,t) + \frac{1}{b}v_x(0,t). \tag{1.4.41}$$

Since v is given by (1.4.37), the right-hand side of (1.4.41) is a known function of t. The solution of (1.4.41) subject to $\phi(0) = 0$ defines $\phi(t)$ uniquely. When this result is used in (1.4.38), we have the solution of (1.4.33).

We work out the details next for the special case $p(x,t) \equiv 0$, $c =$ constant. Thus, according to (1.4.21) (with $\dot{g}(0)$, $g(0^+) = c$) we have

$$v(x,t) = c\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right). \tag{1.4.42}$$

Using the definition (A.2.77) of the complementary error function, we have

$$\frac{d}{dz}\mathrm{erfc}(z) = -\frac{2}{\sqrt{\pi}}e^{-z^2}. \tag{1.4.43}$$

Therefore, $v_x(0,t) = -c/\sqrt{\pi t}$, and (1.4.41) reduces to

$$\dot{\phi} - \frac{a^2}{b^2}\phi = -\frac{ac}{b^2} - \frac{c}{b\sqrt{\pi t}}. \tag{1.4.44}$$

The solution of (1.4.44) subject to $\phi(0) = 0$ is

$$\phi(t) = \frac{c}{a}\left\{1 - e^{a^2t/b^2}\left[1 + \mathrm{erf}\left(\frac{a\sqrt{t}}{b}\right)\right]\right\}. \tag{1.4.45}$$

Thus, $u(x,t)$ is given by (1.4.38) in the form

$$u(x,t) = \frac{c}{a}e^{-ax/b}\left\{1 - e^{-a^2t/b^2}\left[1 + \mathrm{erf}\left(\frac{a\sqrt{t}}{b}\right)\right]\right\} + \frac{c}{b}e^{-ax/b}\int_0^x e^{a\xi/b}\,\mathrm{erfc}\left(\frac{\xi}{2\sqrt{t}}\right)d\xi. \tag{1.4.46}$$

This result can be further simplified by evaluating the integral on the right-hand side. We outline the calculations next, although the final result may be obtained directly using Mathematica or Maple. We have

$$\begin{aligned} I_0 &\equiv \int_0^x e^{a\xi/b}\,\mathrm{erfc}\left(\frac{\xi}{2\sqrt{t}}\right)d\xi \\ &= \frac{b}{a}\left[e^{ax/b}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right) - 1 + \frac{1}{\sqrt{\pi t}}\int_0^x \exp\left(\frac{a\xi}{b} - \frac{\xi^2}{4t}\right)d\xi\right]. \end{aligned}$$

Denoting

$$I_1 \equiv \int_0^x \exp\left(\frac{a\xi}{b} - \frac{\xi^2}{4t}\right)d\xi,$$

we find, upon completing the square in the exponential, that

$$I_1 = e^{a^2t/b^2}\int_0^x \exp\left(\frac{\xi - 2at/b}{2\sqrt{t}}\right)^2 d\xi.$$

The above is a useful trick for integrals with quadratic exponents as in I_1. Now we evaluate I_1 by splitting the integral into two parts as in (1.4.18c) to obtain

$$I_1 = \sqrt{\pi t}e^{a^2t/b^2}\left[\mathrm{erf}\left(\frac{a\sqrt{t}}{b}\right) + \mathrm{erf}\left(\frac{x - 2at/b}{2\sqrt{t}}\right)\right].$$

Therefore, using this result in the expression for I_0 gives

$$I_0 = \frac{b}{a}\left\{e^{ax/b}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right) - 1 + e^{a^2t/b^2}\left[\mathrm{erf}\left(\frac{a\sqrt{t}}{b}\right) + \mathrm{erf}\left(\frac{x - 2at/b}{2\sqrt{t}}\right)\right]\right\}.$$

Now we substitute this expression for I_0 into (1.4.46) to obtain the solution

$$u(x,t) = \frac{c}{a}\left[\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right) - \exp\left(\frac{a^2t}{b^2} - \frac{ax}{b}\right)\mathrm{erfc}\left(\frac{x - 2at/b}{2\sqrt{t}}\right)\right]. \tag{1.4.47}$$

It is a straightforward matter using Mathematica or Maple to verify that (1.4.47) satisfies (1.4.33).

(ii) $a(t)u(0,t) + b(t)u_x(0,t) = c(t)$, *a and b depend on t*

As pointed out earlier, the transformation of dependent variable (1.4.35) is not helpful in this case. Instead, we assume an unknown boundary value for u at $x = 0$,

$$u(0,t) = k(t), \quad t > 0, \tag{1.4.48}$$

where $k(t)$ is as yet unspecified. The solution of the problem consisting of (1.4.33a), (1.4.33b), and (1.4.48) was worked out in (1.4.21). We have

$$u(x,t) = \int_0^t \dot{k}(\tau)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right)d\tau + k(0^+)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right). \tag{1.4.49}$$

We now compute

$$u_x(x,t) = -\frac{1}{\sqrt{\pi}}\int_0^t \frac{\dot{k}(\tau)}{\sqrt{t-\tau}}e^{-x^2/4(t-\tau)}d\tau - \frac{k(0^+)}{\sqrt{\pi t}}e^{-x^2/4t}, \tag{1.4.50}$$

where we have used (1.4.43) to calculate the derivative of the complementary error function. Evaluating (1.4.50) at $x = 0$ and using the result together with (1.4.48) in the boundary condition (1.4.33c) gives the following *integral equation* for $k(t)$:

$$c(t) = a(t)k(t) - \frac{b(t)}{\sqrt{\pi}}\left[\int_0^t \frac{\dot{k}(\tau)}{\sqrt{t-\tau}}d\tau - \frac{k(0^+)}{\sqrt{t}}\right]. \tag{1.4.51}$$

In the first term on the right-hand side of (1.4.50), note the occurrence of the integrable singularity proportional to $(t-\tau)^{-1/2}$ at $\tau = t$. Had we used the form (1.4.22) for the solution u, the corresponding singularity would have been proportional to $x^2(t-\tau)^{-5/2}$, requiring further manipulations to derive a well-behaved result at $x = 0$.

A discussion of techniques for solving the integral equation (1.4.51) is beyond our scope. Once $k(t)$ has been determined, the solution for $u(x,t)$ is given by (1.4.49).

Problems

1.4.1. Verify by direct substitution that (1.4.6) solves (1.4.5), and that (1.4.9) solves (1.4.7).

1.4.2. Verify by direct substitution that (1.4.22) solves (1.4.12).

1.4.3. Consider the linear equation

$$u_t - u_{xx} = 0, \quad 0 \le x, \quad 0 \le t, \tag{1.4.52a}$$

with initial condition

$$u(x,0) = 0, \tag{1.4.52b}$$

and the following boundary condition at $x = 0$:

$$u(0,t) = Ct^n, \quad t > 0, \tag{1.4.52c}$$

where n is a nonnegative constant and C is a positive constant.

As usual, (1.4.52b) implies the boundary condition at infinity

$$u(\infty,t) = 0, \quad t \ge 0. \tag{1.4.52d}$$

a. Use the result (1.4.22) to express the solution in the form

$$u(x,t) = Ct^n f(\theta), \tag{1.4.53}$$

where

$$\theta \equiv \frac{x}{2t^{1/2}} \tag{1.4.54a}$$

and

$$f(\theta) = \frac{2}{\sqrt{\pi}}\int_\theta^\infty \left(1 - \frac{\theta^2}{s^2}\right)^n e^{-s^2}ds. \tag{1.4.54b}$$

b. Show that the similarity form (1.4.53) satisfies (1.4.52), and derive the following differential equation and boundary conditions for $f(\theta)$:

$$f'' + 2\theta f' - 4nf = 0, \tag{1.4.55a}$$

$$f(0) = 1, \tag{1.4.55b}$$

$$f(\infty) = 0. \tag{1.4.55c}$$

Show that the solution of (1.4.55) gives (1.4.54b).

c. Now consider the nonlinear diffusion equation

$$u_t - [k(u)u_x]_x = 0, \quad 0 \le x, \quad 0 \le t, \tag{1.4.56}$$

where $k(u)$ is a prescribed function of u.

The initial condition is (1.4.52b), and the boundary condition at $x = \infty$ is (1.4.52d), whereas at $x = 0$ we have

$$u(0,t) = g(t), \quad t > 0, \tag{1.4.57}$$

for some prescribed function $g(t)$. This problem is discussed in [6].

i. If $k(u) = \lambda u^\nu$, where λ and ν are positive constants, show that the most general $g(t)$ for which a similarity solution exists is

$$g(t) = Ct^n, \tag{1.4.58}$$

where C and n are constants as in (1.4.52c). In this case, the similarity form is

$$u(x,t) = t^n\phi(\zeta), \quad \zeta \equiv \frac{x}{t^{(\nu n+1)/2}}, \tag{1.4.59}$$

and ϕ obeys

$$\lambda\frac{d}{d\zeta}\left(\phi^\nu\frac{d\phi}{d\zeta}\right) + \frac{\nu n+1}{2}\zeta\frac{d\phi}{d\zeta} - n\phi = 0, \tag{1.4.60a}$$

subject to the boundary conditions

$$\phi(0) = C, \quad \phi(\infty) = 0. \tag{1.4.60b}$$

ii. If $k(u)$ is prescribed arbitrarily, show that the most general $g(t)$ for which a similarity solution exists is $g(t) = C$ = constant. In this case the similarity form is

$$u(x,t) = \phi(\theta), \quad \theta = \frac{x}{t^{1/2}}, \tag{1.4.61}$$

and ϕ obeys

$$\frac{d}{d\theta}\left[k(\phi)\frac{d\phi}{d\theta}\right] + \frac{\theta}{2}\frac{d\phi}{d\theta} = 0, \tag{1.4.62a}$$

with boundary conditions

$$\phi(0) = C, \quad \phi(\infty) = 0. \tag{1.4.62b}$$

1.4.4a. Assume that the solution of (1.4.12) on the positive axis may be regarded as the response due to a source of unknown strength $q(t)$ at the origin for an infinite conductor. Therefore, $u(x,t)$ may be expressed in the form (1.3.7) with $p = \delta(x)q(t)$. In this case, (1.3.7) reduces to

$$u(x,t) = \frac{1}{2\sqrt{\pi}}\int_0^t q(\tau)\frac{e^{-x^2/4(t-\tau)}}{\sqrt{t-\tau}}d\tau. \tag{1.4.63}$$

But in order to satisfy the boundary condition (1.4.12c), we must have

$$g(t) = \frac{1}{2\sqrt{\pi}}\int_0^t \frac{q(\tau)d\tau}{\sqrt{t-\tau}}. \tag{1.4.64}$$

This is an integral equation (solved by Abel) for the unknown $q(t)$ in terms of the known $g(t)$.

Use Laplace transforms and the convolution integral to show that

$$q(t) = \frac{2}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{g(\tau)d\tau}{\sqrt{t-\tau}}. \tag{1.4.65}$$

Therefore, the solution of (1.4.12) may also be expressed in the form

$$u(x,t) = \frac{1}{\pi}\int_0^t\left[\frac{e^{-x^2/4(t-\tau)}}{\sqrt{t-\tau}}\frac{d}{d\tau}\int_0^\tau \frac{g(s)ds}{\sqrt{\tau-s}}\right]d\tau. \tag{1.4.66}$$

Show that (1.4.66) reduces to (1.4.22).

b. For the case $g(t) = 1$, we have shown that (1.4.66) reduces to $u(x,t) = \text{erfc}(x/2t^{1/2})$. Suppose that we wish to regard this solution in $0 \le x < \infty$, $0 \le t < \infty$ as being produced by an *unknown initial specification* of u of the form

$$u(x,0) = \begin{cases} 0 & \text{if } x \ge 0, \\ f(x) & \text{if } x < 0, \end{cases} \tag{1.4.67}$$

for the same diffusion equation (1.4.12a) over the infinite interval $-\infty < x < \infty$. With $\tilde{f}(x) = f(-x)$ show that $\tilde{f}$ obeys the integral equation

$$\int_0^\infty \tilde{f}(\xi)e^{-\xi^2/4t}d\xi = 2\sqrt{\pi t}. \tag{1.4.68}$$

Use Laplace transforms to show that $\tilde{f}(\xi) = 2$.

1.4.5a. Modify the calculations leading to (1.4.22) so that you obtain the solution of (1.4.12) with (1.4.12b) replaced by the arbitrary initial condition

$$u(x,0^+) = f(x). \tag{1.4.69}$$

b. Specialize the results in (a) to the case f = constant = u_1, and express the solution in a form such that $u_x(0^+,t)$ is free of singularities. (*Note*: (1.4.22) has an apparent singularity at $x = 0$, whereas (1.4.21) does not.)

1.4.6a. Consider two semi-infinite rods with initial temperatures $u = u_1$ = constant and $u = u_2$ = constant, thermal diffusivities (see (1.1.9)) κ_1^2 = constant and κ_2^2 = constant, and thermal conductivities k_1 = constant and k_2 = constant. Suddenly, at $t = 0$, the two conductors are brought into perfect contact at $x = 0$. Let the first conductor lie on $0 \le x < \infty$ and let the second conductor lie on $-\infty < x \le 0$.

It follows from the integral conservation law (1.1.6) with A = constant that the interface conditions for $t > 0$ are $u(0^+,t) = u(0^-,t)$ and $k_1u_x(0^+,t) = k_2u_x(0^-,t)$. Show this. Use the result in Problem 1.4.5b to show that the heat flow $k_1u_x(0^+,t)$ (or $k_2u_x(0^-,t)$) at the point of contact and $t > 0$ is given by

$$F(t) = \frac{1}{\sqrt{\pi t}}\frac{k_1}{\kappa_1}(u_1 - c), \tag{1.4.70}$$

where c is the constant temperature at $x = 0$:

$$c = \frac{u_2 - \alpha u_1}{1-\alpha}, \quad \alpha = -\frac{k_1\kappa_2}{k_2\kappa_1}. \tag{1.4.71}$$

b. Now consider the situation where these two rods are initially at zero temperature and in perfect thermal contact. Use the method of images to calculate the fundamental solution; that is, solve

$$u_t - \kappa^2 u_{xx} = \delta(t)\delta(x-\xi), \quad 0 < \xi, \tag{1.4.72}$$

on $-\infty < x < \infty$ with $u(x,0^-) = 0$, where $\kappa = \kappa_1$ if $x > 0$ and $\kappa = \kappa_2$ if $x < 0$. Use the interface conditions $u(0^+,t) = u(0^-,t)$ and $k_1u_x(0^+,t) = k_2u_x(0^-,t)$. *Hint*: Assume that in the domain $x < 0$, the solution $u_2(x,t)$ may be regarded as the response to a source of unknown strength B and unknown location ($\xi_1 > 0$) in an infinite medium with the uniform properties κ_2, k_2 *throughout*. Thus, $u_2(x,t)$ corresponds to a "transmitted" temperature due to the primary source at $x = \xi$ and $t = 0$. For the solution $u_1(x,t)$ in the domain $x > 0$, assume that in addition to the response due to the primary source, there is a "reflected" contribution, which may be regarded as the response to an image source of unknown strength A located at the unknown point $x = \xi_2 < 0$ in an infinite rod with properties k_1 and κ_1 throughout. Use the interface conditions to determine A, B, ξ_1, and ξ_2. Verify that in the limits $(k_2/k_1) \to 1$ and $(\kappa_2/\kappa_1) \to 1$, $A \to 0$ and $B \to 1$.

1.4.7. Consider the diffusion equation

$$u_t - u_{xx} = 0, \quad 0 \le t < \infty, \tag{1.4.73}$$

on the *time-dependent* domain $at \le x < \infty$, where a is a constant. We wish to solve the initial- and boundary-value problem having

$$u(x, 0^+) = 0, \tag{1.4.74}$$

$$u(at, t) = g(t), \tag{1.4.75}$$

for $t > 0$ and a prescribed $g(t)$. Thus, u is prescribed as a function of time on the left boundary that moves at a constant speed a.

a. Introduce the transformation of variables $\overline{x} = x - at, \overline{t} = t$ and solve the resulting problem by Laplace transforms.

b. Calculate the appropriate Green's function for the problem in x, t variables and rederive the solution using this.

1.4.8. Use the expression (1.4.25) for G_2 to simplify the solution in (1.4.32a) to the form given by (1.4.32b). Rederive the same result using Laplace transforms.

1.5 Problems in the Finite Domain; Green's Functions

The next step in our development involves problems on the finite domain, which may be taken as the unit interval $0 \le x \le 1$ with no loss of generality (that is, we choose the length L of the domain as the scale to normalize (1.1.9)). As in Section 1.4, we distinguish problems that have $u = 0$ or $u_x = 0$ at either end. Thus, we need to study *four* different Green's functions, and we start with the simplest case.

1.5.1 Green's Function of the First Kind

We refer to the solution satisfying the boundary condition $u = 0$ at both ends as Green's function of the first kind, G_1. More precisely, define the solution of

$$u_t - u_{xx} = \delta(x - \xi)\delta(t - \tau), \quad 0 \le x \le 1, \quad \tau \le t, \tag{1.5.1a}$$

$$u(x, \tau^-) = 0, \tag{1.5.1b}$$

$$u(0, t) = u(1, t) = 0, \quad t > \tau, \tag{1.5.1c}$$

as Green's function $G_1(x, \xi, t - \tau)$. Here, ξ and τ are constants with $0 < \xi < 1$, $0 < \tau$.

Let us construct G_1 using symmetry arguments in terms of appropriate fundamental solutions. Consider the "primary" source $\delta(x - \xi)\delta(t - \tau)$ sketched as $\uparrow$ at the point $x = \xi$, $0 < \xi < 1$, on the unit interval in Figure 1.7.

In order to cancel the contribution of the primary source at the left boundary $x = 0$, we need to introduce a reflected (or image) source of negative unit strength (sketched as $\downarrow$) at the image point $x = -\xi$. This image source must also be turned

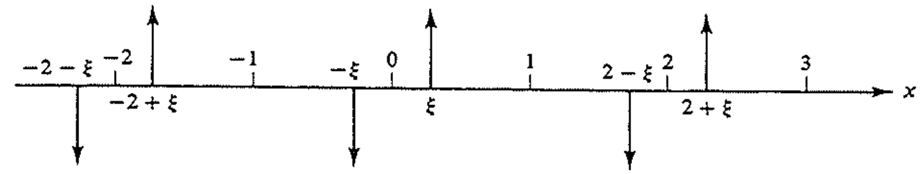

FIGURE 1.7. Primary and reflected sources to give $u = 0$ at $x = 0$ and $x = 1$

on at $t = \tau$. Similarly, to take care of the boundary contribution of the primary source at $x = 2$, we introduce another image source at $x = 2 - \xi$, also turned on at $t = \tau$. But now, the image source at $x = -\xi$ contributes to the boundary value at $x = 1$, and the image source at $x = 2 - \xi$ contributes to the boundary value at $x = 0$. To take care of the first, we introduce the $\uparrow$ unit source at $x = 2 + \xi$. To take care of the second, we introduce the $\uparrow$ unit source at $-2 + \xi$, and so on. The pattern that emerges has positive unit sources at $x = 2n + \xi, n = 0, \pm 1, \pm 2, \ldots$, and negative unit sources are at $x = 2n - \xi, n = 0, \pm 1, \pm 2, \ldots$. The sum of all these source contributions is a representation for Green's function G_1 in the following series form:

$$G_1(x, \xi, t-\tau) \equiv \sum_{n=-\infty}^{\infty} \{F[x-(2n+\xi), t-\tau] - F[x-(2n-\xi), t-\tau]\}, \tag{1.5.2}$$

where F is defined in (1.2.20).

Green's function G_1 has the interesting symmetry property

$$G_1(x, \xi, t - \tau) \equiv G_1(\xi, x, t - \tau). \tag{1.5.3}$$

The corresponding steady-state result is noted in Appendix A.1.3. To demonstrate this symmetry property, we note that the right-hand side of (1.5.3) is by definition given by

$$G_1(\xi, x, t-\tau) = \sum_{n=-\infty}^{\infty} [F(\xi - 2n - x, t - \tau) - F(\xi - 2n + x, t - \tau)]. \tag{1.5.4}$$

Since F is an even function of its first argument, we can rewrite the first term in the summation as $F(-\xi + 2n + x, t - \tau)$. Furthermore, since the summation ranges over $-\infty < n < \infty$, the infinite sum of these terms remains the same if we replace n by $-n$. Therefore, we may write

$$G_1(\xi, x, t-\tau) = \sum_{n=-\infty}^{\infty} [F(-\xi - 2n + x, t - \tau) - F(\xi - 2n + x, t - \tau)], \tag{1.5.5}$$

which is just $G_1(x, \xi, t - \tau)$.

In terms of heat conduction, the result (1.5.3) is intuitively obvious and physically consistent. Suppose we consider a conductor with uniform properties and with its two endpoints maintained at the same temperature, here normalized to be zero. Fix any two distinct locations x and ξ on the conductor and carry out the following two experiments. In the first experiment we turn on a unit source of heat at time τ at the point ξ and measure the temperature at the point x and time $t > \tau$. This gives the result $G_1(x, \xi, t-\tau)$ for the measured temperature. In the second experiment, we reverse the locations of the source and observer without changing the values of τ or t and find that the temperature at ξ, given by $G_1(\xi, x, t-\tau)$, is the same as that measured in the first experiment.

Using G_1 and superposition, we can now solve the inhomogeneous problem (see (1.4.5))

$$u_t - u_{xx} = p(x,t), \quad 0 \le x \le 1, \quad 0 \le t, \tag{1.5.6a}$$

with zero initial condition

$$u(x, 0^-) = 0 \tag{1.5.6b}$$

and zero boundary conditions at both ends,

$$u(0,t) = u(1,t) = 0 \quad \text{for} \quad t > 0, \tag{1.5.6c}$$

in the form

$$u(x,t) = \int_0^t d\tau \int_0^1 [p(\xi,\tau)G_1(x,\xi,t-\tau)]d\xi. \tag{1.5.7}$$

Similarly, as in (1.4.7)–(1.4.8), we solve the problem with $p(x,t) = 0$ and nonzero initial condition

$$u(x, 0^+) = f(x), \tag{1.5.8}$$

instead of (1.5.6b), in the form

$$u(x,t) = \int_0^1 f(\xi)G_1(x,\xi,t)d\xi. \tag{1.5.9}$$

Green's functions for the remaining three homogeneous boundary-value problems are listed in Problem 1.5.2.

1.5.2 Connection with Separation of Variables

You may be wondering how the result in (1.5.9) is related to the solution we obtain by the more conventional separation of variables approach that is usually discussed in a first course in partial differential equations. We explore this question next. (Problem 1.5.6 gives a review of the basic ideas of separation of variables.)

To solve

$$u_t - u_{xx} = 0, \tag{1.5.10a}$$

$$u(x, 0^+) = f(x), \tag{1.5.10b}$$

$$u(0,t) = u(1,t) = 0, \quad t > 0, \tag{1.5.10c}$$

we assume that u can be expressed in the "separated" form:

$$u(x,t) = X(x)T(t). \tag{1.5.11}$$

Substituting (1.5.11) into (1.5.10a) gives

$$X\dot{T} - X''T = 0, \quad \text{or} \quad \frac{\dot{T}}{T} = \frac{X''}{X}, \tag{1.5.12}$$

where the dot indicates d/dt and the double prime indicates d^2/dx^2. The second part of (1.5.12) can hold only if it equals a constant, and we quickly convince ourselves that this constant must be negative, say $-\lambda^2$. (Why?)

So, we obtain the *eigenvalue problem*

$$X'' + \lambda^2 X = 0, \quad X(0) = X(1) = 0, \tag{1.5.13}$$

associated with (1.5.10). The solution is the *eigenfunction*

$$X_n = b_n \sin \lambda_n x, \quad \lambda_n = n\pi,$$

where b_n is arbitrary and n is an integer. Thus, the solution of (1.5.10) in a series of eigenfunctions is just the *Fourier sine series*

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin n\pi x. \tag{1.5.14}$$

Substituting (1.5.14) into (1.5.10a), or using $\dot{T}_n + \lambda_n^2 T_n = 0$, gives $B_n = c_n e^{-n^2\pi^2 t}$, where c_n = constant.

To determine the c_n, we impose the initial condition (1.5.10b) and make use of orthogonality to obtain

$$c_n = 2\int_0^1 f(\xi) \sin n\pi\xi \, d\xi. \tag{1.5.15}$$

Thus, the solution of (1.5.10) may be written in series form as

$$u(x,t) = \sum_{n=1}^{\infty} \left[2\int_0^1 f(\xi)\sin n\pi\xi \, d\xi \right] e^{-n^2\pi^2 t} \sin n\pi x.$$

If we interchange summation and integration (a step that is nearly never questioned in a course in applied mathematics!), we obtain

$$u(x,t) = \int_0^1 f(\xi)H(x,\xi,t)d\xi, \tag{1.5.16a}$$

where

$$H(x,\xi,t) \equiv 2\sum_{n=1}^{\infty} (\sin n\pi\xi)e^{-n^2\pi^2 t} \sin n\pi x. \tag{1.5.16b}$$

Comparing (1.5.16) with (1.5.9) shows that these two results do not look alike. In fact, in order for the two results to agree, we must be able to show that $G_1 = H$. This is indeed the case, and is a consequence of a certain identity for the *theta* *function*. For example, see page 75 of [12]. It is instructive to work out this identity in detail next.

We may use trigonometric identities to rewrite H in the form

$$H(x,\xi,t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}\cos n\pi(x-\xi) - \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}\cos n\pi(x+\xi). \tag{1.5.17}$$

Now, the expression for G_1 in (1.5.2) agrees with (1.5.17) if we can show that

$$\sum_{n=-\infty}^{\infty} F(x+\xi-2n,t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}\cos n\pi(x+\xi) \tag{1.5.18a}$$

and

$$\sum_{n=-\infty}^{\infty} F(x-\xi-2n,t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-n^2\pi^2 t}\cos n\pi(x-\xi). \tag{1.5.18b}$$

These two conditions are equivalent and reduce to the simple condition

$$\frac{1}{\sqrt{\eta}}\sum_{n=-\infty}^{\infty} e^{-\pi(z-n)^2/\eta} = \sum_{n=-\infty}^{\infty} e^{-n^2\pi\eta}\cos 2n\pi z, \tag{1.5.19}$$

if we write $(x+\xi)$ or $(x-\xi)$ as $2z$, set $\eta = \pi t$, and use the expression (1.2.20) defining F.

Denote $\sqrt{\eta}$ times the expression on the left-hand side of (1.5.19) by ϕ; that is,

$$\phi(\eta,z) \equiv \sum_{n=-\infty}^{\infty} e^{-\pi(z-n)^2/\eta}. \tag{1.5.20}$$

Clearly, ϕ is an even function of z (that is, $\phi(\eta,-z) = \phi(\eta,z)$). Also, it is periodic in z with unit period: $\phi(\eta,z+1) = \phi(\eta,z)$. Therefore, we may expand ϕ in a Fourier cosine series:

$$\phi(\eta,z) = \sum_{\nu=-\infty}^{\infty} \alpha_\nu(\eta)\cos 2\pi\nu z, \tag{1.5.21a}$$

where

$$\alpha_\nu(\eta) = \int_0^1 \sum_{n=-\infty}^{\infty} e^{-\pi(\zeta-n)^2/\eta}\cos 2\pi\nu\zeta\, d\zeta. \tag{1.5.21b}$$

Interchanging integration and summation in (1.5.21b) gives

$$\alpha_\nu(\eta) = \sum_{n=-\infty}^{\infty}\int_0^1 e^{-\pi(\zeta-n)^2/\eta}\cos 2\pi\nu\zeta\, d\zeta. \tag{1.5.22}$$

Now change the integration variable and let $s = n-\zeta$ to obtain

$$\alpha_\nu(\eta) = \sum_{n=-\infty}^{\infty}\int_{n-1}^{n} e^{-\pi s^2/\eta}\cos 2\pi\nu s ds$$

$$= \int_{-\infty}^{\infty} e^{-\pi s^2/\eta}\cos 2\pi\nu s\, ds = \sqrt{\eta}e^{-\pi\nu^2\eta}. \tag{1.5.23}$$

Thus, we have proven the identity

$$\sum_{n=-\infty}^{\infty} e^{-\pi(z-n)^2/\eta} = \sum_{n=-\infty}^{\infty} \sqrt{\eta}e^{-\pi n^2\eta}\cos 2\pi nz, \tag{1.5.24}$$

which is (1.5.19) when we divide by $\sqrt{\eta}$.

In conclusion, the series representation for G_1 converges to the same result as the series for H, even though these series *do not agree term by term*. This latter observation means that if we truncate the series for G_1, the resulting approximation will be valid in a different sense than the approximation obtained by truncating the Fourier series H. Let us pursue this idea further, as it will provide a useful characterization of the two approaches we have used.

Consider first what happens if we truncate the series (1.5.2) at $n = N$ for G_1. Clearly, we are neglecting all the heat sources located at distances greater than $2N+\xi$ on the positive axis and greater than $2N-\xi$ on the negative axis. For short times, the response due to these sources is very small over the unit interval (because we are ignoring only the weak exponential tails of the corresponding F functions). Thus, the Green's function representation (1.5.9), when G_1 is truncated for some $n = N$, *should be valid for short times*. In particular, the boundary conditions at $x = 0$ and $x = 1$ are only approximately satisfied with the truncated series, and this approximation deteriorates as t gets large. On the other hand, if we truncate the Fourier series representation (1.5.16), the boundary conditions are *exactly* satisfied for all times, but the initial condition will be described only approximately. Thus, the truncated series (1.5.16) should provide a *good approximation for t large*. A more careful analysis of the convergence properties of the G_1 and H series confirms the above intuitive conclusions.

We reiterate that both expressions converge to the same solution if the infinite series are summed. We shall see in Chapter 3 in examples for the wave equation that this property of Green's functions versus eigenfunction expansions is also true there. It is a useful result, as we are able to have an approximation involving a finite number of terms for both t small and t large.

1.5.3 Connection with Solution by Laplace Transforms

A third approach for solving the problem in (1.5.10) is to use Laplace transforms with respect to t. For simplicity, consider the special case $f = 1$. Using the notation $U(x,s)$ for the Laplace transform of $u(x,t)$ (see Section A.2.6), we obtain

$$U_{xx} - sU = -1, \quad U(0,s) = U(1,s) = 0.$$

The solution is easily obtained in the form

$$U(x,s) = \frac{1}{s(e^{\sqrt{s}} - e^{-\sqrt{s}})}[e^{\sqrt{s}} - e^{-\sqrt{s}} + (e^{-\sqrt{s}}-1)e^{\sqrt{s}x} - (e^{\sqrt{s}}-1)e^{-\sqrt{s}x}]. \tag{1.5.25}$$

The solution for $u(x, t)$ is then given by the inversion integral (A.2.41b); that is,

$$u(x,t) = \frac{1}{2\pi i}\int_{0^+-i\infty}^{0^++i\infty} e^{st}U(x,s)ds. \tag{1.5.26}$$

Note the branch points at $s = 0$ and $s = \infty$. Since we must choose the branch of $\sqrt{s}$ that is positive when s is along the positive real axis, it is convenient to cut the s-plane along the negative real axis; hence $c = 0^+$ for the vertical contour.

The expression (1.5.26) cannot be evaluated in terms of a finite number of elementary functions. One standard approximation for a Laplace transform inversion is the "large s" approximation, which consists of expanding (1.5.25) in series form for s large and then integrating the result, term by term, in (1.5.26). As discussed in texts on complex variables (for example, see page 279 of [8]), this gives an approximation for $u(x, t)$ valid for t *small*.

To see this, just change the variables in (1.5.26), setting $s = \sigma/t$ and consider the limit $|s| \to \infty$, $|\sigma|$ *fixed*. Clearly, this implies that we need to take $t \to 0$, and in effect, the substitution σ/t for s in $U(x, s)$ accomplishes this.

If we expand the denominator of (1.5.25) and take the product of this series with the numerator, we find that U equals the particular solution $1/s$ plus four series in the form

$$\begin{aligned} U(x,s) = &\frac{1}{s} + \frac{1}{s}\left[e^{-\sqrt{s}(2-x)} + e^{-\sqrt{s}(4-x)} + e^{-\sqrt{s}(6-x)} + \dots\right] \\ &- \frac{1}{s}\left[e^{-\sqrt{s}x} + e^{-\sqrt{s}(2+x)} + e^{-\sqrt{s}(4+x)} + \dots\right] \\ &- \frac{1}{s}\left[e^{-\sqrt{s}(1-x)} + e^{-\sqrt{s}(3-x)} + e^{-\sqrt{s}(5-x)} + \dots\right] \\ &+ \frac{1}{s}\left[e^{-\sqrt{s}(1+x)} + e^{-\sqrt{s}(3+x)} + e^{-\sqrt{s}(5+x)} + \dots\right]. \end{aligned}$$

These series can be rearranged in the form

$$U(x,s) = \frac{1}{s} + \frac{1}{s}\sum_{n=1}^{\infty}(-1)^n e^{-\sqrt{s}(n-x)} - \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n e^{-\sqrt{s}(n+x)}. \tag{1.5.27}$$

Using (1.5.26) or tables of Laplace transforms, we find that the transform of

$$f(t) = \operatorname{erfc}\left(\frac{\lambda}{2\sqrt{t}}\right), \tag{1.5.28a}$$

with $t > 0$ and λ real, is

$$F(s) = \frac{1}{s}e^{-\lambda\sqrt{s}}. \tag{1.5.28b}$$

Therefore, the termwise inversion of (1.5.27) gives the series

$$u(x,t) = 1 + \sum_{n=1}^{\infty}(-1)^n \operatorname{erfc}\left(\frac{n-x}{2\sqrt{t}}\right) - \sum_{n=0}^{\infty}(-1)^n \operatorname{erfc}\left(\frac{n+x}{2\sqrt{t}}\right). \tag{1.5.29}$$

It is left as an exercise (Problem 1.5.4) to show that this series is the same as the one resulting from the Green's function representation (1.5.9) when we take $f = 1$ and integrate the series for G_1 term by term. This gives a confirmation of our earlier intuitive argument that the truncated Green's function representation of the solution is valid for t small.

At any rate, the *exact* expressions (1.5.9), (1.5.16a) with $f = 1$, and (1.5.26) define the *same* function $u(x, t)$. The advantage of (1.5.9) and (1.5.16a) over (1.5.26) is that these are in terms of real quadratures, whereas (1.5.26) is a complex integral. Another example of the use of Laplace transforms to calculate the solution of the diffusion equation in a bounded domain is given in Problem 1.5.5.

1.5.4 Uniqueness of Solutions

In this section we show that solutions of the initial- and boundary-value problem for the diffusion equation are unique. We consider solutions of

$$u_t - u_{xx} = 0,\quad 0 \le x \le 1,\quad 0 \le t < \infty, \tag{1.5.30a}$$

with initial condition

$$u(x, 0^+) = f(x), \tag{1.5.30b}$$

and one of the following four boundary conditions:

$$u(0,t) = g(t),\quad u(1,t) = h(t), \tag{1.5.31a}$$

$$u(0,t) = g(t),\quad u_x(1,t) = h(t), \tag{1.5.31b}$$

$$u_x(0,t) = g(t),\quad u(1,t) = h(t), \tag{1.5.31c}$$

$$u_x(0,t) = g(t),\quad u_x(1,t) = h(t). \tag{1.5.31d}$$

Here g and h are arbitrarily prescribed in each case.

In preparation for this proof, we first derive an integral identity for solutions of (1.5.30a). Multiply (1.5.30a) by $u(x, t)$ and integrate the result with respect to x on the unit interval to obtain

$$\int_0^1 uu_t dx = \int_0^1 uu_{xx}dx.$$

Since the interval is independent of t, we may write the left-hand side of this expression as $(d/dt)\int_0^1(u^2/2)dx$, and integrating the right-hand side by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2(x,t)dx = uu_x\Big|_0^1 - \int_0^1 u_x^2(x,t)dx. \tag{1.5.32}$$

The identity (1.5.32) is true for any solution of (1.5.30a). Suppose that u_1 and u_2 are two solutions of (1.5.30a), each of which satisfies the initial condition (1.5.30b) and one of the four pairs of boundary conditions (1.5.31). If we denote the difference by $u_1 - u_2 \equiv v(x,t)$, then $v(x,t)$ satisfies the problem

$$v_t - v_{xx} = 0,$$
$$v(x,0) = 0,$$
$$vv_x = 0 \quad \text{at } x = 0 \quad \text{and} \quad x = 1.$$

Therefore, the identity (1.5.32) for v becomes

$$\frac{1}{2}\frac{d}{dt}\int_0^1 v^2(x,t)dx = -\int_0^1 v_x^2(x,t)dx \le 0.$$

Or if we let

$$I(t) \equiv \frac{1}{2}\int_0^1 v^2(x,t)dx \ge 0$$

and

$$G(t) \equiv -\int_0^1 v_x^2(x,t)dx \le 0,$$

we have

$$\frac{dI}{dt} = G(t), \quad \text{that is,} \quad I(t) - I(0) = \int_0^t G(\tau)d\tau \le 0. \tag{1.5.33}$$

Thus, $I(t) - I(0) \le 0$. But $I(0) = 0$; hence, $I(t) \le 0$. According to its definition, $I(t) \ge 0$. So, we must have $I(t) \equiv 0$, and the integral of a nonnegative quantity, such as v^2, can vanish only if $v(x,t) = 0$. Thus, we have proven that $u_1(x,t) = u_2(x,t)$.

1.5.5 Inhomogeneous Boundary Conditions

As discussed in Section 1.4, we can transform a homogeneous equation with inhomogeneous boundary conditions to an inhomogeneous equation with homogeneous boundary conditions. To illustrate the idea, consider (1.5.30a) with initial condition (1.5.30b) and boundary conditions (1.5.31a).

To homogenize the boundary conditions, assume a transformation of dependent variable $u \to w$ in the following form that is linear in x,

$$u(x,t) \equiv w(x,t) + \alpha(t)x + \beta(t), \tag{1.5.34}$$

with as yet unspecified functions α and β of the time, to be chosen such that the boundary conditions for the resulting problem for w are homogeneous.

Using (1.5.34), we compute

$$u_t = w_t + \dot{\alpha}x + \dot{\beta},$$
$$u_x = w_x + \alpha, \quad u_{xx} = w_{xx}.$$

Therefore,

$$u_t - u_{xx} = w_t + \dot{\alpha}x + \dot{\beta} - w_{xx} = 0,$$

that is, w obeys the inhomogeneous problem

$$w_t - w_{xx} = -\dot{\alpha}x - \dot{\beta}.$$

In order to have $w(0,t) = 0$, we find from (1.5.34) that we must set $\beta(t) = g(t)$. Similarly, in order to have $w(1,t) = 0$, we must set $\alpha(t) = h(t) - g(t)$. Thus, the transformation relation is

$$u(x,t) \equiv w(x,t) + x[h(t) - g(t)] + g(t), \tag{1.5.35}$$

and w obeys the inhomogeneous equation

$$w_t - w_{xx} = [\dot{g}(t) - \dot{h}(t)]x - \dot{g}(t) \equiv p(x,t) \tag{1.5.36a}$$

subject to the initial condition

$$w(x,0) = f(x) - x[h(0^+) - g(0^+)] - g(0^+) \equiv q(x) \tag{1.5.36b}$$

and homogeneous boundary conditions

$$w(0,t) = w(1,t) = 0. \tag{1.5.36c}$$

The solution of the problem (1.5.36) is just the sum of the solutions (1.5.7) and (1.5.9) with $f = q$; that is,

$$w(x,t) = \int_0^t d\tau \int_0^1 p(\xi,\tau)G_1(x,\xi,t-\tau)d\xi + \int_0^1 q(\xi)G_1(x,\xi,t)d\xi. \tag{1.5.37}$$

Having found $w(x,t)$, we obtain $u(x,t)$ from (1.5.35). Note that the form (1.5.36) is also appropriate for a solution using Fourier series, as homogeneous boundary conditions are also crucial in being able to superpose eigensolutions. Problem 1.5.7 concerns the solution for the case (1.5.31b).

Problems

1.5.1a. Show that Green's function for the following general homogeneous boundary-value problem for the steady-state diffusion equation

$$-u'' = \delta(x-\xi); \quad 0 \le x \le 1, \quad 0 < \xi < 1, \tag{1.5.38}$$
$$u(0) + a_0u'(0) = 0; \quad a_0 = \text{constant}, \tag{1.5.39a}$$
$$u(1) + a_1u'(1) = 0; \quad a_1 = \text{constant}, \tag{1.5.39b}$$

is given by

$$G(x,\xi) = \begin{cases} \frac{(1-\xi+a_1)(x-a_0)}{1+a_1-a_0}; & x < \xi, \\ \frac{(1-x+a_1)(\xi-a_0)}{1+a_1-a_0}; & x > \xi. \end{cases} \tag{1.5.40}$$

Give a physical reason why G becomes infinite if $a_0 - a_1 = 1$.

b. Give a physical reason why Green's function for (1.5.38) with the homogeneous boundary conditions $u'(0) = u'(1) = 0$ does not exist.

1.5.2. Use symmetry arguments to show that Green's function for the diffusion equation

$$u_t - u_{xx} = \delta(t-\tau)\delta(x-\xi) \tag{1.5.41}$$

with zero initial condition and each of the following three types of homogeneous boundary conditions is given in the specified form.

a. $u(0,t) = u_x(1,t) = 0$ has

$$G_2(x,\xi,t-\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \{F[x-(2n+\xi), t-\tau] - F[x-(2n-\xi), t-\tau]\}. \tag{1.5.42a}$$

b. $u_x(0,t) = u(1,t) = 0$ has

$$G_3(x,\xi,t-\tau) = \sum_{n=-\infty}^{\infty} (-1)^n \{F[x-(2n+\xi), t-\tau] + F[x-(2n-\xi), t-\tau]\}. \tag{1.5.42b}$$

c. $u_x(0,t) = u_x(1,t) = 0$ has

$$G_4(x,\xi,t-\tau) = \sum_{n=-\infty}^{\infty} \{F[x-(2n+\xi), t-\tau] + F[x-(2n-\xi), t-\tau]\}. \tag{1.5.42c}$$

What symmetry properties, if any, can you uncover for G_2, G_3, and G_4 if $x \to \xi, \xi \to x$?

d. Use the results of parts (a)–(c) to solve the general initial/boundary value problem for

$$u_t - u_{xx} = p(x,t), \tag{1.5.43a}$$

$$u(x,0) = f(x), \tag{1.5.43b}$$

and each of the following pairs of boundary conditions for $t > 0$ after introducing an appropriate homogenizing transformation as in Section 1.5.5

$$u(0,t) = g_1(t);\quad u_x(1,t) = g_2(t), \tag{1.5.44a}$$

$$u_x(0,t) = h_1(t);\quad u(1,t) = h_2(t), \tag{1.5.44b}$$

$$u_x(0,t) = h_1(t);\quad u_x(1,t) = g_2(t). \tag{1.5.44c}$$

1.5.3. Evaluate (1.5.7) for the special case where $p = \delta(x-\zeta)$, where ζ is a fixed constant on $0 < \zeta < 1$. Show that as $t \to \infty$, your result reduces to Green's function for the steady-state problem derived in Appendix A.1 (see (A.1.40)).

1.5.4. Evaluate (1.5.9) for $f = 1$ and show that the resulting series is the same as (1.5.29).

1.5.5a. Show that the Laplace transform $U(x,s)$ of the solution of

$$u_t - u_{xx} = 0;\quad 0 \le x \le 1;\quad 0 \le t, \tag{1.5.45a}$$

$$u(0,t) = 1;\quad u(1,t) = 0, \tag{1.5.45b}$$

$$u(x,0) = 0, \tag{1.5.45c}$$

is

$$U(x,s) = \frac{1}{s}\,\frac{\sinh\sqrt{s}(1-x)}{\sinh\sqrt{s}}. \tag{1.5.46}$$

b. Rewrite (1.5.46) in the form

$$U(x,s) = \frac{1}{s}\cdot\frac{1}{(1-e^{-2\sqrt{s}})}(e^{-\sqrt{s}x} - e^{-\sqrt{s}(2-x)}), \tag{1.5.47}$$

and expand the factor

$$\frac{1}{1-e^{-2\sqrt{s}}} = \sum_{n=0}^{\infty} e^{-2n\sqrt{s}} \tag{1.5.48}$$

for large s to obtain the series

$$U(x,s) = \frac{1}{s}\sum_{n=0}^{\infty}(e^{-\sqrt{s}(2n+x)} - e^{-\sqrt{s}(2n+2-x)}). \tag{1.5.49}$$

Now use (1.5.28) to show that the solution $u(x,t)$ has the series form

$$u(x,t) = \sum_{n=0}^{\infty}\left[\operatorname{erfc}\left(\frac{2n+x}{2\sqrt{t}}\right) - \operatorname{erfc}\left(\frac{2n+2-x}{2\sqrt{t}}\right)\right]. \tag{1.5.50}$$

c. Calculate the solution of (1.5.45) using Green's function and superposition after homogenizing the boundary condition at $x = 0$. Show that this result agrees with (1.5.50).

1.5.6. This is a review problem to illustrate separation of variables and Fourier series. Consider

$$u_t - u_{xx} = x\sin t,\quad 0 \le x \le 1,\quad 0 \le t, \tag{1.5.51a}$$

$$u(x,0) = x(1-x), \tag{1.5.51b}$$

$$u(0,t) = u_x(1,t) = 0 \quad \text{if } t > 0. \tag{1.5.51c}$$

a. Look for a solution of the homogeneous equation (1.5.51a) in the separated form $u(x,t) = X(x)T(t)$, and show that X is given by any one of the eigenfunctions

$$X_n(x) = \alpha_n \sin\lambda_n x, \tag{1.5.52}$$

where the eigenvalues are $\lambda_n = (2n+1)\pi/2$ for $n = 0, 1, 2, \ldots$ and $\alpha_n =$ constant.

b. Based on this result assume a solution of (1.5.51) in the form of a series of eigenfunctions:

$$u(x,t) = \sum_{n=0}^{\infty} A_n(t) \sin \lambda_n x, \tag{1.5.53}$$

where the $A_n(t)$ are functions of t to be specified. Also, expand the right-hand side of (1.5.51a) in a series of the eigenfunctions X_n,

$$x \sin t = \left(\sum_{n=0}^{\infty} b_n \sin \lambda_n x \right) \sin t. \tag{1.5.54}$$

Use orthogonality to show that $b_n = 8(-1)^n/\pi^2(2n+1)^2$.

Now substitute (1.5.52) into (1.5.51a) with (1.5.54) for its right-hand side to show that the $A_n(t)$ satisfy

$$\frac{dA_n}{dt} + \lambda_n^2 A_n = b_n \sin t. \tag{1.5.55}$$

c. Solve (1.5.55) to obtain

$$A_n(t) = A_n(0)e^{-\lambda_n^2 t} + \frac{b_n}{\lambda_n^2 + 1}(\lambda_n^2 \sin t - \cos t + e^{-\lambda_n^2 t}). \tag{1.5.56}$$

d. Use (1.5.53) with $A_n(t)$ given by (1.5.56) in the initial condition (1.5.51b) to obtain

$$A_n(0) = \frac{32 - 8\pi(-1)^n(2n+1)}{\pi^3(2n+1)^3}. \tag{1.5.57}$$

1.5.7. Solve

$$u_t - u_{xx} = p(x,t), \quad 0 \le x \le 1, \quad 0 \le t, \tag{1.5.58a}$$

$$u(x, 0^+) = f(x), \tag{1.5.58b}$$

$$u(0,t) = g(t), \quad u_x(1,t) = h(t), \tag{1.5.58c}$$

using Green's function as well as separation of variables after having transformed to a homogeneous boundary-value problem.

1.6 Higher-Dimensional Problems

The diffusion equation in two or more space dimensions is given by the following dimensionless form of (1.1.18):

$$u_t - \Delta u = p, \tag{1.6.1}$$

where Δ is the Laplace operator and p is a prescribed function of the spatial variables and the time. For certain domains where one or more of the coordinates are bounded, solutions may be calculated using separation of variables. This technique may also be combined with Fourier transforms with respect to one or more coordinates that have an unbounded range. An example is outlined in Problem 1.6.4b.

For a complete discussion of separation of variables see [22]. Another approach for solving (1.6.1) is to take its Laplace transform with respect to t. The result is a Helmholtz equation for the transformed variable, and this equation is discussed in Chapter 2. See Section 2.3.2 and Problems 2.3.4, 2.6.1, and 2.6.2. Here we will only consider solutions using Green's functions, and we begin our discussion with a derivation of the fundamental solution.

1.6.1 *The Fundamental Solution*

Consider the n-dimensional diffusion equation with a unit source turned on at the origin at time $t = 0$:

$$u_t - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = \delta(t)\delta(x_1)\delta(x_2)\ldots\delta(x_n), \tag{1.6.2}$$

where n is a positive integer. As in (1.2.6)–(1.2.7), we have the zero initial condition

$$u(x_1, \ldots, x_n, 0^-) = 0, \tag{1.6.3}$$

and require u to vanish at infinity:

$$u(x_1, \ldots, u_n, t) \to 0 \quad \text{as} \quad r = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \to \infty. \tag{1.6.4}$$

In view of the fact that the source term on the right-hand side of (1.6.2) produces a spherically symmetric solution, we need only consider the spherically symmetric Laplacian, and (1.6.2) has the form

$$u_t - u_{rr} - \frac{(n-1)}{r} u_r = \delta(t)\delta_n(r). \tag{1.6.5}$$

We have used the notation $\delta_n(r)$ to denote the n-dimensional delta function

$$\delta_n(r) = \delta(x_1)\delta(x_2)\ldots\delta(x_n). \tag{1.6.6}$$

Consider the volume integral in terms of the Cartesian coordinates $x_1, \ldots, x_n$ of the n-dimensional delta function over some domain D in this n-dimensional space. By simply applying the properties of the one-dimensional delta function to each of the n integrals defining the volume integral, we have the following generalization of the definition for the one-dimensional case

$$\int_D \cdots \int \delta(x_1)\delta(x_2)\ldots\delta(x_n)dx_1dx_2\ldots dx_n = \begin{cases} 1 & \text{if the origin is in } D, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6.7}$$

For $n \ge 2$, we may also write (1.6.7) in terms of the n-dimensional delta function $\delta_n(r)$ and the appropriate volume element dV

$$\int_D \cdots \int \delta_n(r)dV = \begin{cases} 1 & \text{if } r = 0 \text{ is in } D, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6.8}$$

For example, if $n = 2$ and D is the interior of a circle of radius ϵ centered at the origin, then $dV = r\,dr\,d\theta$, and we have

$$\int_{r=0}^{\epsilon}\int_{\theta=0}^{2\pi} \delta_2(r) r\, d\theta\, dr = 2\pi \int_0^{\epsilon} r\delta_2(r)dr = 1, \tag{1.6.9a}$$

where r and θ are polar coordinates in the plane: $x = r\cos\theta$, $y = r\sin\theta$. If $n = 3$ and D is the interior of a sphere of radius ϵ centered at the origin, the corresponding result is

$$\int_{r=0}^{\epsilon}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \delta_3(r) r^2 \sin\phi\, d\phi\, d\theta\, dr = 4\pi \int_0^{\epsilon} r^2\delta_3(r)dr = 1, \tag{1.6.9b}$$

where r, θ, and ϕ are the spherical polar coordinates: $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\sin\phi$. More generally, for an n-dimensional sphere of radius ϵ centered at the origin, (1.6.8) reduces to

$$\omega_n \int_{r=0}^{\epsilon} r^{n-1}\delta_n(r)dr = 1, \tag{1.6.9c}$$

where ω_n is the "area" of the n-dimensional *unit* sphere.

To calculate ω_n consider the following identity:

$$\int_{-\infty}^{\infty}\!\!\cdots\!\int e^{-x_1^2-x_2^2-\cdots-x_n^2} dx_1 dx_2 \ldots dx_n = \int_0^{\infty} e^{-r^2} r^{n-1}\omega_n dr, \tag{1.6.10}$$

where $r^2 = x_1^2 + \ldots + x_n^2$. The left-hand side of (1.6.10) is just $\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^n = \pi^{n/2}$. The right-hand side is

$$\omega_n \int_0^{\infty} e^{-r^2} r^{n-1} dr = \frac{\omega_n}{2}\int_0^{\infty} e^{-\sigma}\sigma^{\frac{n}{2}-1} d\sigma = \frac{\omega_n}{2}\Gamma\left(\frac{n}{2}\right), \tag{1.6.11}$$

where $\Gamma(z)$ is the *gamma function* defined by

$$\Gamma(z) \equiv \int_0^{\infty} e^{-\sigma}\sigma^{z-1} d\sigma;\ z > 0. \tag{1.6.12}$$

Therefore, the area ω_n of the n-dimensional unit sphere is

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{1.6.13}$$

Coming back to (1.6.5), we solve the homogeneous problem using similarity. Proceeding as in Section 1.2.1, we find that the fundamental solution $F(r, t)$ has the following similarity structure (cf. (1.2.15))

$$\alpha^n F(\alpha r, \alpha^2 t) = F(r, t) \tag{1.6.14}$$

for any positive constant α. Setting

$$F(r, t) = t^{-n/2} f(\theta),\ \theta = rt^{-1/2}, \tag{1.6.15a}$$

we see that (1.6.15a) satisfies (1.6.14) and gives the following ordinary differential equation for f:

$$f'' + \left(\frac{\theta}{2} + \frac{n-1}{\theta}\right) f' + \frac{n}{2} f = 0.$$

It is easily seen that

$$f = Ce^{-\theta^2/4},\ C = \text{constant},$$

is a solution that upon substitution into (1.6.15) gives

$$F(r, t) = \frac{C}{t^{n/2}} e^{-r^2/4t}. \tag{1.6.15b}$$

This solution has the appropriate singularity at $r = 0, t = 0$ and decays as $r \to \infty$, $t > 0$ or $t \to \infty$, $r > 0$. The other solution of (1.6.15) gives an unbounded contribution to the total heat content in the domain as in the one-dimensional case.

To evaluate C, we integrate (1.6.5) over the entire n-dimensional space D_∞ to obtain

$$\int_{D_\infty}\!\!\cdots\!\int F_t dV = \int_{D_\infty}\!\!\cdots\!\int \Delta F dV + \int_{D_\infty}\!\!\cdots\!\int \delta(t)\delta_n(r) dV. \tag{1.6.16}$$

Using Cartesian coordinates we have

$$\int_{D_\infty}\!\!\cdots\!\int \Delta F dV \equiv \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\frac{\partial^2 F}{\partial x_1^2} + \ldots + \frac{\partial^2 F}{\partial x_n^2}\right] dx_1 \ldots dx_n = 0, \tag{1.6.17}$$

because for each $i = 1, \ldots, n$, $\partial F/\partial x_i$ vanishes if any one of its arguments x_j equals $\pm\infty$. The second integral on the right-hand side of (1.6.16) gives $\delta(t)$, and we obtain

$$\int_{D_\infty}\!\!\cdots\!\int F_t dV = \frac{d}{dt}\int_{D_\infty}\!\!\cdots\!\int F dV = \delta(t).$$

Therefore,

$$\int_{D_\infty}\!\!\cdots\!\int F dV = 1 \quad \text{if} \quad t > 0, \tag{1.6.18}$$

as in the one-dimensional case (cf. (1.2.19)). Using the result (1.6.15b) for F in (1.6.18) gives

$$\frac{C}{t^{n/2}}\int_0^{\infty} e^{-r^2/4t}\omega_n r^{n-1} dr = 1. \tag{1.6.19a}$$

Changing the integration variable from r to $s = r^2/4t$ gives

$$2^{n-1}\omega_n C \int_0^{\infty} e^{-s} s^{\frac{n}{2}-1} ds = 2^{n-1}\omega_n C\Gamma\left(\frac{n}{2}\right) = 1. \tag{1.6.19b}$$

Therefore,

$$C = \frac{1}{2^{n-1}\omega_n \Gamma(n/2)} = \frac{1}{2^n \pi^{n/2}}, \qquad (1.6.20)$$

and the fundamental solution is

$$F(r, t) = \frac{1}{2^n \pi^{n/2} t^{n/2}} e^{-r^2/4t}. \qquad (1.6.21)$$

More generally, the fundamental solution at time t at a point P with coordinates $x_1, \ldots, x_n$ due to a source located at the point Q with coordinates $\xi_1, \ldots, \xi_n$ and turned on at time τ is

$$F(r_{PQ}, t - \tau) = \frac{e^{-r_{PQ}^2/4(t-\tau)}}{2^n \pi^{n/2} (t-\tau)^{n/2}}, \qquad (1.6.22)$$

where we have introduced the notation

$$r_{PQ} = \sqrt{\sum_{i=1}^{n} (x_i - \xi_i)^2} \qquad (1.6.23)$$

for the distance between the observer at P and the source point Q.

1.6.2 Initial-Value Problem in the Infinite Domain

Consider the general initial-value problem for the three-dimensional diffusion equation in the infinite domain:

$$u_t - u_{xx} - u_{yy} - u_{zz} = p(x, y, z, t), \qquad (1.6.24a)$$

$$u(x, y, z, 0) = f(x, y, z). \qquad (1.6.24b)$$

Here p and f are prescribed, and $p = 0$ if $t < 0$. The corresponding one-dimensional problem was discussed in Section 1.3. The basic ideas are the same; we split (1.6.24) into two problems as in (1.3.2) and solve each using the fundamental solution. The result is

$$u(x, y, z, t) = \int_{\tau=0}^{\infty} \int_{\xi=-\infty}^{\infty} \int_{\eta=-\infty}^{\infty} \int_{\zeta=-\infty}^{\infty} F(r_{PQ}, t - \tau) p(\xi, \eta, \zeta, \tau) d\zeta\, d\eta\, d\xi\, d\tau$$
$$+ \int_{\xi=-\infty}^{\infty} \int_{\eta=-\infty}^{\infty} \int_{\zeta=-\infty}^{\infty} f(\xi, \eta, \zeta) F(r_{PQ}, t) d\zeta\, d\eta\, d\xi, \qquad (1.6.25)$$

where $r_{PQ}^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2$, and F is given by (1.6.22) with $n = 3$. Similar formulas can be written down for the solution for any n.

(i) *Example, axisymmetric problem in two dimensions*

Consider the axisymmetric problem in two dimensions

$$u_t - \left(u_{rr} + \frac{1}{r} u_r\right) = 0, \qquad (1.6.26a)$$

$$u(r, 0) = f(r). \qquad (1.6.26b)$$

The fundamental solution is ($n = 2$, $r_{PQ}^2 = (x - \xi)^2 + (y - \eta)^2$)

$$F(r_{PQ}, t - \tau) = \frac{1}{4\pi(t - \tau)} \exp\left(-\frac{(x - \xi)^2 + (y - \eta)^2}{4(t - \tau)}\right). \qquad (1.6.27)$$

We introduce the polar coordinates r, θ for P defined by $x = r\cos\theta$, $y = r\sin\theta$ and set $\xi = \rho\cos\phi$, $\eta = \rho\sin\phi$. Then $r_{PQ}^2 = r^2 - \rho^2 - 2r\rho\cos(\theta - \phi)$, and (1.6.27) becomes

$$F(r_{PQ}, t - \tau) = \frac{1}{4\pi(t - \tau)} \exp\left(-\frac{r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)}{4(t - \tau)}\right). \qquad (1.6.28)$$

In the superposition integral corresponding to (1.6.25) only the second term contributes, and we use polar coordinates to obtain

$$u(r, t) = \frac{e^{-r^2/4t}}{4\pi t} \int_0^{\infty} e^{-\rho^2/4t} \rho f(\rho) \left[\int_0^{2\pi} e^{(r\rho\cos\phi)/2t} d\phi\right] d\rho. \qquad (1.6.29)$$

The definite integral with respect to ϕ can be evaluated explicitly. For any positive constant α, we have

$$\int_0^{2\pi} e^{\alpha\cos\phi} d\phi = 2\pi I_0(\alpha), \qquad (1.6.30)$$

where I_0 is the modified Bessel function of order zero. Therefore, (1.6.29) simplifies to

$$u(r, t) = \frac{e^{-r^2/4t}}{2t} \int_0^{\infty} f(\rho) \rho e^{-\rho^2/4t} I_0\left(\frac{r\rho}{2t}\right) d\rho. \qquad (1.6.31)$$

1.6.3 Green's Function for Various Simple Domains

The use of image sources to satisfy boundary conditions also generalizes to higher-dimensional problems for certain simple geometries. Three planar examples are discussed next to illustrate ideas.

(i) *The half-plane $y \geq 0$ with $u(x, 0, t) = 0$*

Consider

$$u_t - (u_{xx} + u_{yy}) = \delta(x - \xi)\delta(y - \eta)\delta(t - \tau) \qquad (1.6.32a)$$

in the upper half-plane: $-\infty < x < \infty$, $0 \leq y \leq \infty$ for positive constants ξ, η, τ. The initial condition is

$$u(x, y, \tau^-) = 0, \qquad (1.6.32b)$$

and boundary conditions are

$$u(x, 0, t) = u(x, \infty, t) = 0. \qquad (1.6.32c)$$

Using a negative image source at $x = \xi$, $y = -\eta$, $t = \tau$ we obtain Green's function using (1.6.22) with $n = 2$

$$G(x-\xi, y, \eta, t-\tau) = \frac{1}{4\pi(t-\tau)}\left[e^{-r_{PQ}^2/4(t-\tau)} - e^{-r_{P\overline{Q}}^2/4(t-\tau)}\right], \quad (1.6.33)$$

where

$$r_{PQ}^2 = (x-\xi)^2 + (y-\eta)^2 \quad (1.6.34a)$$
$$r_{P\overline{Q}} = (x-\xi)^2 + (y+\eta)^2. \quad (1.6.34b)$$

Knowing G, we can solve the general initial- and boundary-value problem in the upper half-plane,

$$u_t - u_x - u_{yy} = p(x, y, t), \quad (1.6.35a)$$
$$u(x, y, 0) = f(x, y), \quad (1.6.35b)$$
$$u(x, 0, t) = g(x, t), \ t > 0, \quad (1.6.35c)$$

for prescribed functions p, f, and g using Green's function and superposition after the boundary condition (1.6.35c) is homogenized. The details are entirely analogous to the one-dimensional case discussed in Section 1.4 and are left as an exercise (Problem 1.6.1).

The same ideas can be used to compute Green's function in the half-space in three dimensions and to construct Green's function of the second kind where the normal derivative of u vanishes along the boundary.

(ii) *The quarter-plane* $x \geq 0$, $y \geq 0$ *with* $u(x, 0, t) = u(0, y, t) = 0$

Green's function satisfies

$$u_t - (u_{xx} + u_{yy}) = \delta(x-\xi)\delta(y-\eta)\delta(t-\tau), \quad (1.6.36a)$$
$$u(x, y, \tau^-) = 0, \quad (1.6.36b)$$
$$u(x, 0, t) = 0, \ x > 0, \quad (1.6.36c)$$
$$u(0, y, t) = 0, \ y > 0. \quad (1.6.36d)$$

For this domain, the positive primary source of unit strength is located at $x = \xi > 0$, $y = \eta > 0$ and turned on at time $t = \tau$. In order to have $u = 0$ on both the positive x- and y-axes we need to introudce negative image sources of unit strength at the points $x = \xi$, $y = -\eta$ and $x = -\xi$, $y = \eta$. We also need a positive image source of unit strength at $x = -\xi$, $y = -\eta$. This maintains the symmetry relative to the two coordinate axes.

Therefore, the solution is given by

$$G(x, \xi, y, \eta, t-\tau) = \frac{1}{4\pi(t-\tau)}\left[e^{-r_1^2/4(t-\tau)} - e^{-r_2^2/4(t-\tau)} - e^{-r_3^2/4(t-\tau)} + e^{-r_4^2/4(t-\tau)}\right], \quad (1.6.37)$$

where

$$r_1^2 = r_{PQ} = (x-\xi)^2 + (y-\eta)^2, \quad (1.6.38a)$$
$$r_2^2 = (x-\xi)^2 + (y+\eta)^2, \quad (1.6.38b)$$
$$r_3^2 = (x+\xi)^2 + (y-\eta)^2, \quad (1.6.38c)$$
$$r_4^2 = (x+\xi)^2 + (y+\eta)^2. \quad (1.6.38d)$$

Using (1.6.37) we can solve the general initial- and boundary-value problem in the quarter-plane. See Problem 1.6.2. Problem 1.6.3 concerns the solution in the quarter plane with $u_y(x, 0, t)$ prescribed.

The symmetry idea also generalizes to corner domains in higher dimensions, e.g., $x \geq 0$, $y \geq 0$, $z \geq 0$ in three dimensions.

(iii) *The infinite strip* $0 \leq y \leq 1$, $-\infty < x < \infty$ *with* $u(x, 0, t) = u(x, 1, t) = 0$

Green's function of the first kind for this domain satisfies

$$u_t - u_{xx} - u_{yy} = \delta(x-\xi)\delta(y-\eta)\delta(t-\tau), \quad (1.6.39a)$$
$$u(x, y, \tau^-) = 0, \quad (1.6.39b)$$
$$u(x, 0, t) = u(x, 1, t) = 0, \ t > 0. \quad (1.6.39c)$$

The solution of (1.6.39) is entirely analogous to the one- dimensional version (1.5.2), and we have

$$G(x-\xi, y, \eta, t-\tau) = \sum_{n=-\infty}^{\infty} \{F(r_n, t-\tau) - F(\overline{r}_n, t-\tau)\}, \quad (1.6.40)$$

where

$$F(r, t) = \frac{e^{-r^2/4t}}{2\pi t}, \quad (1.6.41)$$
$$r_n^2 = (x-\xi)^2 + [y-(2n+\eta)]^2, \quad (1.6.42a)$$
$$\overline{r}_n^2 = (x-\xi)^2 + [y-(2n-\eta)]^2. \quad (1.6.42b)$$

We can now use (1.6.40) to solve the general initial- and boundary -value problem in the infinite strip. See Problem 1.6.4a. This problem can also be solved by Fourier transforms with respect to x followed by separation of variables as discussed in Problem 1.6.4b.

Boundary-value problems where u_y is specified on $y = 0$ or $y = 1$ or both can also be solved using the appropriate Green's function as in Problem 1.5.2.

Problems

1.6.1. Use the homogenizing transformation $w(x, y, t) = u(x, y, t) - g(x, t)$ to show that if u solves (1.6.35), then w is the solution of

$$w_t - w_{xx} - w_{yy} = h(x, t) + \delta(t)k(x, y), \quad (1.6.43a)$$

$$w(x, y, 0^-) = 0, \quad (1.6.43b)$$
$$w(x, 0, t) = 0, \quad t > 0, \quad (1.6.43c)$$

where

$$h(x, t) \equiv p(x, y, t) - g_t(x, t) + g_{xx}(x, t), \quad (1.6.44a)$$
$$k(x, y) \equiv f(x, y) - g(x, 0). \quad (1.6.44b)$$

Using Green's function (1.6.33), the solution of (1.6.35) then becomes

$$u(x, y, t) = \int_0^t \int_{-\infty}^{\infty} \int_0^{\infty} h(\xi, \tau) G(x - \xi, y, \eta, t - \tau) d\eta\, d\xi\, d\tau + \int_{-\infty}^{\infty} \int_0^{\infty} k(\xi, \eta) G(x - \xi, y, \eta, t) d\eta\, d\xi + g(x, t). \quad (1.6.45)$$

Develop the result in (1.6.45) using (1.4.18) to obtain (see (1.4.21))

$$u(x, y, t) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} \operatorname{erf}\left(\frac{y}{2\sqrt{t-\tau}}\right) \times \left[\int_{-\infty}^{\infty} h(\xi, \tau) e^{-(x-\xi)^2/4(t-\tau)} d\xi\right] d\tau - \frac{1}{2\sqrt{\pi t}} \operatorname{erf}\left(\frac{y}{2\sqrt{t}}\right) \int_{-\infty}^{\infty} g(\xi, 0) e^{-(x-\xi)^2/4t} d\xi + \frac{1}{4\pi t} \int_{-\infty}^{\infty} \int_0^{\infty} f(\xi, \eta) \left[e^{-[(x-\xi)^2 + (y-\eta)^2]/4t} - e^{-[(x-\xi)^2 + (y+\eta)^2]/4t}\right] d\eta\, d\xi. \quad (1.6.46)$$

1.6.2. Consider the general initial- and boundary-value problem in the quarter plane $x \geq 0$, $y \geq 0$:

$$u_t - u_{xx} - u_{yy} = p(x, y, t), \quad (1.6.47a)$$
$$u(x, y, 0) = f(x, y), \quad (1.6.47b)$$
$$u(x, 0, t) = g_1(x, t), \quad (1.6.47c)$$
$$u(0, y, t) = g_2(y, t). \quad (1.6.47d)$$

Introduce the homogenizing transformation

$$w(x, y, t) = u(x, y, t) - \alpha(x, y, t), \quad (1.6.48)$$

where α is a function that satisfies $\alpha(x, 0, t) = g_1(x, t)$, $\alpha(0, y, t) = g_2(y, t)$. For example, we may choose

$$\alpha(x, y, t) = \frac{g_1(x, t)x}{\sqrt{x^2 + y^2}} + \frac{g_2(y, t)y}{\sqrt{x^2 + y^2}}. \quad (1.6.49)$$

Show that if u satisfies (1.6.47), then w is governed by

$$w_t - w_{xx} - w_{yy} = h(x, y, t) + \delta(t)k(x, y), \quad (1.6.50a)$$
$$w(x, y, 0^-) = 0, \quad (1.6.50b)$$
$$w(x, 0, t) = w(0, y, t) = 0, \quad t > 0, \quad (1.6.50c)$$

where

$$h(x, y, t) \equiv p(x, y, t) - \alpha_t(x, y, t) + \alpha_{xx}(x, y, t) + \alpha_{yy}(x, y, t), \quad (1.6.51a)$$
$$k(x, y) \equiv f(x, y) - \alpha(x, y, 0). \quad (1.6.51b)$$

Solve (1.6.50) using Green's function (1.6.37).

1.6.3. What is Green's function for the corner domain $0 \leq x < \infty, 0 \leq y < \infty$ with boundary conditions $u(0, y, t) = 0$, $u_y(x, 0, t) = 0$? Use this result to calculate the solution of (1.6.47), where we replace (1.6.47c) by $u_y(x, 0, t) = g_3(x, t)$.

1.6.4a. Consider the diffusion equation in two dimensions in the infinite strip $-\infty < x < \infty$, $0 \leq y \leq 1$ with prescribed source distribution, and initial and boundary values for u given by

$$u_t - u_{xx} - u_{yy} = p(x, y, t), \quad (1.6.52a)$$
$$u(x, y, 0) = f(x, y), \quad (1.6.52b)$$
$$u(x, 0, t) = g_1(x, t), \quad t > 0, \quad (1.6.52c)$$
$$u(x, 1, t) = g_2(x, t), \quad t > 0. \quad (1.6.52d)$$

Introduce the homogenizing transformation

$$w(x, y, t) = u(x, y, t) + (y - 1)g_1(x, t) - yg_2(x, t) \quad (1.6.53)$$

to show that w satisfies

$$w_t - w_{xx} - w_{yy} = h(x, y, t) + \delta(t)k(x, y), \quad (1.6.54a)$$
$$w(x, y, 0^-) = 0, \quad (1.6.54b)$$
$$w(x, 0, t) = w(x, 1, t) = 0, \quad t > 0, \quad (1.6.54c)$$

where

$$h(x, y, t) \equiv p(x, y, t) + (1 - y)(g_{1_t} - g_{1_{xx}}) - y(g_{2_t} - g_{2_{xx}}), \quad (1.6.55)$$
$$k(x, y) \equiv f(x, y) + (y - 1)g_1(x, 0) - yg_2(x, 0). \quad (1.6.56)$$

Calculate the solution of (1.6.54) using Green's function (1.6.40).

b. An alternative approach for solving (1.6.52) is to take Fourier transforms with respect to x. Show that the transformed variable $\overline{u}(k, y, t)$ satisfies (overbars indicate the Fourier transform, see (A.2.9a))

$$\bar{u}_t - \bar{u}_{yy} + k^2\bar{u} = \bar{p}(k, y, t), \quad (1.6.57a)$$
$$\bar{u}(k, y, 0) = \bar{f}(k, y), \quad (1.6.57b)$$
$$\bar{u}(k, 0, t) = \bar{g}_1(k, t), \ t > 0, \quad (1.6.57c)$$
$$\bar{u}(k, 1, t) = \bar{g}_2(k, t), \ t > 0. \quad (1.6.57d)$$

In preparation for solving (1.6.57) by separation of variables, introduce the homogenizing transformation (1.6.53),

$$\bar{w}(k, y, t) = \bar{u}(k, y, t) + (y - 1)\bar{g}_1(k, t) - y\bar{g}_2(k, t), \quad (1.6.58)$$

and show that $\bar{w}$ satisfies

$$\bar{w}_t - \bar{w}_{yy} + k^2\bar{w} = \bar{p} + (y-1)(\bar{g}_{1_t} + k^2\bar{g}_1) - y(\bar{g}_{2_t} + k^2\bar{g}_2) \equiv q(k, y, t), \quad (1.6.59a)$$

$$\bar{w}(k, y, 0) = \bar{f}(k, y) + (y - 1)\bar{g}_1(k, 0) - y\bar{g}_2(k, 0) \equiv r(k, y), \quad (1.6.59b)$$

$$\bar{w}(k, 0, t) = \bar{w}(k, 1, t) = 0, \ t > 0. \quad (1.6.59c)$$

Solve (1.6.59) by separation of variables in the form

$$\bar{w}(k, y, t) = \sum_{n=1}^{\infty} B_n(k, t) \sin n\pi y, \quad (1.6.60)$$

where

$$B_n(k, t) = \left[B_n(k, 0) + \int_0^t q_n(k, \tau) e^{(n^2\pi^2+k^2)\tau} d\tau \right] e^{-(n^2\pi^2+k^2)t}, \quad (1.6.61)$$

and $B_n(k, 0)$, $q_n(k, t)$ are the Fourier coefficients

$$B_n(k, 0) = 2\int_0^1 r(k, y) \sin n\pi y \, dy, \quad (1.6.62a)$$

$$q_n(k, t) = 2\int_0^1 q(k, y, t) \sin n\pi y \, dy. \quad (1.6.62b)$$

1.7 Burgers' Equation

The quasilinear diffusion equation

$$u_t + uu_x - \epsilon u_{xx} = 0, \ \epsilon > 0, \quad (1.7.1)$$

is attributed to Burgers, who in 1948 proposed it as a mathematical model for turbulence [7]. Actually, (1.7.1) was first studied by Bateman in 1915 in modeling the motion of a fluid with small viscosity ϵ [5]. Although (1.7.1) may be obtained as a limiting form of the x-component of the momentum equation for viscous

flows, as first shown in [32], it does not model turbulence. Nevertheless, (1.7.1) is a fundamental *evolution equation* that arises in a number of unrelated applications where viscous and nonlinear effects are equally important. Examples are discussed in [16] and in Section 6.2.5 of [26]. This equation also models traffic flow and is derived in Section 5.1.2.

Hopf [24], and Cole [9] independently showed that (1.7.1) may be transformed to the linear diffusion equation of this chapter. We now work out this transformation and discuss how it may be used to solve initial- and boundary-value problems for (1.7.1).

1.7.1 The Cole–Hopf Transformation

This transformation of dependent variable $u \to v$ is defined by

$$u \equiv -2\epsilon \frac{v_x}{v}. \quad (1.7.2)$$

We then calculate

$$u_t = -2\epsilon \frac{v_{xt}}{v} + 2\epsilon \frac{v_x v_t}{v^2},$$
$$u_x = -2\epsilon \frac{v_{xx}}{v} + 2\epsilon \frac{v_x^2}{v^2},$$
$$u_{xx} = -2\epsilon \frac{v_{xxx}}{v} + 6\epsilon \frac{v_x v_{xx}}{v^2} - \frac{4\epsilon v_x^3}{v^3}.$$

Substituting these expressions into (1.7.1) gives

$$\frac{v_x}{v}(\epsilon v_{xx} - v_t) - (\epsilon v_{xx} - v_t)_x = 0. \quad (1.7.3)$$

Thus, *any* solution $v(x, t)$ of (1.7.3), when used in (1.7.2), gives an expression $u(x, t)$, that satisfies (1.7.1).

In particular, if v satisfies the diffusion equation

$$\epsilon v_{xx} - v_t = 0, \quad (1.7.4)$$

it also solves (1.7.3) trivially, and the resulting $u(x, t)$ satisfies (1.7.1).

Although the parameter ϵ may be transformed out of (1.7.1) (and hence also out of (1.7.4)) by an appropriate scaling of the x and t variables, it is more instructive to retain it in the solution because we can then study how the results behave in the limit $\epsilon \to 0$. This is a singular perturbation problem that we will discuss in Section 8.2.3.

1.7.2 Initial-Value Problem on $-\infty < x < \infty$

Let us study how we can use the preceding result to solve the initial-value problem for Burgers' equation:

$$u_t + uu_x - \epsilon u_{xx} = 0, \ -\infty < x < \infty, \quad (1.7.5a)$$
$$u(x, 0) = f(x). \quad (1.7.5b)$$

According to (1.7.2), the new variable $v(x,t)$ must initially satisfy

$$f(x) = -\frac{2\epsilon v_x(x,0)}{v(x,0)}. \tag{1.7.6}$$

This is a linear first-order ordinary differential equation for $v(x,0)$ and has the general solution

$$v(x,0) = \alpha e^{(-1/2\epsilon)\int_0^x f(s)ds} \equiv \alpha g(x), \ \alpha = \text{constant}. \tag{1.7.7}$$

Thus, for a given $f(x)$, we compute $g(x)$ by quadrature. Of course, it is understood that the integral $\int_0^x f(s)ds$ exists. So, we need to solve the following *linear problem* for $v(x,t)$:

$$v_t - \epsilon v_{xx} = 0, \ -\infty < x < \infty, \tag{1.7.8a}$$

$$v(x,0) = \alpha g(x). \tag{1.7.8b}$$

This is essentially (1.3.3) and has the solution (1.3.9) after replacing $u \to v$, $f \to \alpha g, t \to \epsilon t$:

$$v(x,t) = \frac{\alpha}{\sqrt{4\pi\epsilon t}}\int_{-\infty}^{\infty} g(\xi)e^{-(x-\xi)^2/4\epsilon t}d\xi. \tag{1.7.9}$$

It then follows that

$$v_x(x,t) = \frac{-\alpha}{\sqrt{4\pi\epsilon t}}\int_{-\infty}^{\infty} \frac{g(\xi)(x-\xi)}{2\epsilon t}e^{-(x-\xi)^2/4\epsilon t}d\xi. \tag{1.7.10}$$

Therefore, using (1.7.2) to compute $u(x,t)$ gives

$$u(x,t) = \frac{\int_{-\infty}^{\infty} g(\xi)\frac{(x-\xi)}{t}e^{-(x-\xi)^2/4\epsilon t}d\xi}{\int_{-\infty}^{\infty} g(\xi)e^{-(x-\xi)^2/4\epsilon t}d\xi}, \tag{1.7.11}$$

in which the constant α cancels out.

We shall use these results in discussing discontinuous solutions of the first-order equation

$$u_t + uu_x = 0 \tag{1.7.12}$$

in Chapter 5. We compute (1.7.11) explicitly for the case where $f(x)$ is piecewise constant in Problem 1.7.1.

1.7.3 Boundary-Value Problems

The solution of Burgers' equation on the semi-infinite or bounded interval in x is more complicated than the solution we have derived in (1.7.11). We now consider some special cases.

(i) *Semi-infinite interval:* $0 \leq x < \infty$

The problem is

$$u_t + uu_x - \epsilon u_{xx} = 0, \ 0 \leq x < \infty, \tag{1.7.13a}$$

$$u(x,0) = f(x), \tag{1.7.13b}$$

$$u(0,t) = h(t), \ t > 0. \tag{1.7.13c}$$

Using (1.7.2) we obtain the following problem for the new dependent variable $v(x,t)$

$$v_t - \epsilon v_{xx} = 0, \tag{1.7.14a}$$

$$v(x,0) = \alpha \exp\left(-\frac{1}{2\epsilon}\int_0^x f(s)ds\right) \equiv \alpha g(x), \tag{1.7.14b}$$

$$h(t)v(0,t) + 2\epsilon v_x(0,t) = 0. \tag{1.7.14c}$$

If h = constant and $f = 0$, (1.7.14b) and (1.7.14c) reduce to

$$v(x,0) = \alpha = \text{constant}, \tag{1.7.15a}$$

$$hv(x,0) + 2\epsilon v_x(x,0) = 0, \ h = \text{constant}. \tag{1.7.15b}$$

In (1.7.14b) and (1.7.15a), the constant α is arbitrary.

To use previously calculated results, we set

$$\overline{v} = v - \alpha, \ \overline{t} = \epsilon t, \ \overline{x} = x$$

to obtain

$$\overline{v}_{\overline{t}} - \overline{v}_{\overline{x}\overline{x}} = 0, \tag{1.7.16a}$$

$$\overline{v}(\overline{x},0) = 0, \tag{1.7.16b}$$

$$h\overline{v}(\overline{x},0) + 2\epsilon\overline{v}_{\overline{x}}(\overline{x},0) = -h\alpha. \tag{1.7.16c}$$

The solution is given by (1.4.47) with $u \to \overline{v}$, $a \to h$, $b \to 2\epsilon$, $c = -h\alpha$, $x \to \overline{x}$, $t \to \overline{t}$,

$$\overline{v}(\overline{x},t) = -\alpha\left[\operatorname{erfc}\left(\frac{\overline{x}}{2\sqrt{\overline{t}}}\right) - \exp\left(\frac{h^2\overline{t}}{4\epsilon^2} - \frac{h\overline{x}}{2\epsilon}\right)\operatorname{erfc}\left(\frac{\overline{x} - 2h\overline{t}/2\epsilon}{2\sqrt{\overline{t}}}\right)\right] \tag{1.7.17a}$$

or

$$v(x,t) = \alpha\left[1 - \operatorname{erfc}\left(\frac{x}{2\sqrt{\epsilon t}}\right) + \exp\left(\frac{h^2t}{4\epsilon} - \frac{hx}{2\epsilon}\right)\operatorname{erfc}\left(\frac{x - ht}{2\sqrt{\epsilon t}}\right)\right]. \tag{1.7.17b}$$

We now use this result to evaluate (1.7.2) for $u(x,t)$ and obtain

$$u(x,t) = h\frac{\operatorname{erfc}\left(\frac{x-ht}{2\sqrt{\epsilon t}}\right)}{\exp\left(\frac{hx}{2\epsilon} - \frac{h^2t}{4}\right)\operatorname{erf}\left(\frac{x}{2\sqrt{\epsilon t}}\right) + \operatorname{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right)}. \tag{1.7.18}$$

In [31], this result is attributed to J.D. Cole.

If h is not constant, the above approach does not apply, but we may use the idea discussed in Section 1.4.7 of replacing (1.7.14c) by an unknown boundary value

$v(0, t) = k(t)$, then deriving an integral equation for $k(t)$. The details are entirely analogous to those discussed in Section 1.4.7. See Problem 1.7.2.

(ii) *Finite interval* $0 \le x \le \pi$

The following initial- and boundary-value problem for Burgers' equation is discussed in [9]:

$$u_t + uu_x = \epsilon u_{xx}, \quad 0 \le x \le \pi, \tag{1.7.19a}$$

$$u(x, 0) = f(x), \tag{1.7.19b}$$

$$u(0, t) = u(\pi, t) = 0, \quad t > 0. \tag{1.7.19c}$$

The problem for $v(x, t)$ defined by (1.7.2) satisfies

$$v_t - \epsilon v_{xx} = 0, \tag{1.7.20a}$$

$$v(x, 0) = \alpha g(x), \quad \alpha = \text{constant}, \tag{1.7.20b}$$

$$v_x(0, t) = v_x(\pi, t) = 0, \quad t > 0, \tag{1.7.20c}$$

where α is arbitrary and $g(x)$ is defined in (1.7.14b).

The solution for $v(x, t)$ is easily derived using separation of variables in the form

$$v(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 \epsilon t} \cos nx, \tag{1.7.21}$$

where

$$a_n = \frac{2\alpha}{\pi} \int_0^{\pi} g(x) \cos nx \, dx. \tag{1.7.22}$$

The transformation relation (1.7.2) gives the solution of (1.7.19) in the form

$$u(x, t) = 2\epsilon \frac{\sum_{n=1}^{\infty} n a_n e^{-n^2 \epsilon t} \sin nx}{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 \epsilon t} \cos nx}. \tag{1.7.23}$$

A discussion of the qualitative features of the solution is given in [9]. The problem where $f(x)$ has a discontinuity in the interval $0 \le x \le 1$ is discussed in [29]. This problem is of interest in understanding the long-term behavior of a shock layer for Burgers' equation. We still study shock layers in Chapter 5.

Problems

1.7.1. Consider Burgers' equation on $-\infty < x < \infty$,

$$u_t + uu_x = \epsilon u_{xx}. \tag{1.7.24}$$

a. For the piecewise constant initial condition

$$u(x, 0) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0, \end{cases} \tag{1.7.25}$$

derive the solution in the form

$$u(x, t) = \frac{e^{-x/\epsilon} \operatorname{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right) - \operatorname{erfc}\left(-\frac{x+t}{2\sqrt{\epsilon t}}\right)}{e^{-x/\epsilon} \operatorname{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right) + \operatorname{erfc}\left(-\frac{x+t}{2\sqrt{\epsilon t}}\right)}. \tag{1.7.26}$$

b. For the piecewise constant initial condition

$$u(x, 0) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases} \tag{1.7.27}$$

derive the solution in the form

$$u(x, t) = \frac{-\operatorname{erfc}\left(\frac{x+t}{2\sqrt{\epsilon t}}\right) + e^{-x/\epsilon} \operatorname{erfc}\left(\frac{t-x}{2\sqrt{\epsilon t}}\right)}{\operatorname{erfc}\left(\frac{x+t}{2\sqrt{\epsilon t}}\right) + e^{-x/\epsilon} \operatorname{erfc}\left(\frac{t-x}{2\sqrt{\epsilon t}}\right)}. \tag{1.7.28}$$

1.7.2. To study the problem (1.7.13) for Burgers' equation, replace (1.7.14c) with the boundary condition

$$v(0, t) = k(t), \tag{1.7.29}$$

where $k(t)$ is as yet unspecified. Use the results in Sections 1.4.2–1.4.3 to write the solution for $v(x, t)$ in terms of the unknown $k(t)$ in the form

$$\begin{aligned} v(x, t) = {} & \frac{\alpha}{2\sqrt{\pi \epsilon t}} \int_0^{\infty} g(\xi) \left[e^{-(x-\xi)^2/4\epsilon t} - e^{-(x+\xi)^2/4\epsilon t} \right] d\xi \\ & + \int_0^t \dot{k}(\tau) \operatorname{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau \\ & + k(0^+) \operatorname{erfc}\left(\frac{x}{2\sqrt{\epsilon t}}\right). \end{aligned} \tag{1.7.30}$$

Use the condition (1.7.14c) to derive the following integral equation for $k(t)$ (Assume $\alpha = k(0^+)$)

$$h(t)k(\tau) = \phi(t) + 2\sqrt{\frac{\epsilon}{\pi}} \int_0^t \frac{\dot{k}(\tau)}{\sqrt{t-\tau}} d\tau, \tag{1.7.31}$$

where

$$\phi(t) = 2k(0^+)\sqrt{\frac{\epsilon}{\pi t}} \left[1 - \int_0^{\infty} g(\xi) e^{-\xi^2/4\epsilon t} d(\xi^2/4\epsilon t) \right]. \tag{1.7.32}$$

Note that $\phi(t) = 0$ if $f(x) = 0$.