Some Jargons for PDEs

- Homogeneous equation:

$$u_t - u_{xx} = 0. \tag{1}$$

- Inhomogeneous equation:

$$u_t - u_{xx} = g(x,t), \tag{2}$$

  where g is a *known* function, representing a heat source/sink.

- Inhomogeneous Dirichlet boundary conditions:

$$u(0,t) = g(t). \tag{3}$$

- Inhomogeneous Neumann boundary conditions:

$$u_x(0,t) = g(t). \tag{4}$$

- Homogeneous boundary condition:

$$u(0,t) = 0 \text{ or } u_x(0,t) = 0. \tag{5}$$

Variations on the theme of 1-D Heat Diffusion

(Semi-)Infinite domain

- Time-dependent
  - Dirichlet B.C.
    - Homogeneous boundary-value problems with zero or non-zero IC *(we already covered it.)*
      Sec 4-15: Instantaneous Heating or Cooling, Sec. 4-16: Cooling of the Oceanic Lithosphere.
    - Inhomogeneous boundary-value problems *(case study 1)*.
      Sec 4-14: Periodic Heating.
  - Neumann B.C. *(case study 2)*
    Sec 4-26: Heating or Cooling by a Constant Surface Heat Flux.
- Steady state $\rightarrow$ special (and much simpler!) cases of the corresponding time-dependent type.
  Sec 4-6 to 4-12.

Finite domain *(case study 3)*

- Once you figure out the Green's function, the procedure to get a solution is the same.
  Sec 4-17: Plate cooling model of the oceanic lithosphere.

Case Study 1/3

- The full set of equation:

$$u_t - u_{xx} = 0,\ 0 \le x < \infty,\ 0 \le t < \infty, \tag{6}$$
$$u(0,t) = g(t),\ u(\infty,t) = 0, \tag{7}$$
$$u(x,0) = 0. \tag{8}$$

- Recall that the fundamental solution and the Green's function for the semi-infinite domain were derived for a homogeneous boundary value problem (BVP). We put a negative image source to enforce the boundary condition!
- So, we need to perform ***homogenizing transformation*** in order to utilize them in the current inhomogeneous BVP.
- We define a new dependent variable (i.e., a function for the temperature field) as

$$w(x,t) \equiv u(x,t) - g(t). \tag{9}$$

Case Study 1/3

- We can easily see that w obeys the inhomogeneous equation with homogeneous BC:

$$w_t - w_{xx} = -\dot{g}(t),\ 0 \leq x < \infty,\ 0 \leq t < \infty, \tag{10}$$
$$w(0,t) = 0,\ w(\infty,t) = -g(t), \tag{11}$$
$$w(x,0) = -g(0). \tag{12}$$

  - Note that the condition at $x = \infty$ doesn't affect the image source technique.
  - This problem is equivalent to

$$w_t - w_{xx} = -\dot{g}(t) - g(0)\delta(t), \tag{13}$$
$$w(0,t) = 0, \tag{14}$$
$$w(x,0) = 0,\ t > 0. \tag{15}$$

Case Study 1/3

- The solution in the general form is

$$w(x,t) = \int_0^t \int_0^\infty \frac{-\dot{g}(\tau)}{2\sqrt{\pi(t-\tau)}} \left[e^{-(x-\xi)^2/4(t-\tau)} - e^{-(x+\xi)^2/4(t-\tau)} \right] d\xi d\tau$$
$$- \int_0^\infty \frac{g(0)}{2\sqrt{\pi t}} \left[e^{-(x-\xi)^2/4t} - e^{-(x+\xi)^2/4t} \right] d\xi. \quad (16)$$

- This solution involves two definite integrals

$$I = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(x-\xi)^2/4(t-\tau)}}{2\sqrt{t-\tau}} d\xi \quad (17)$$

and

$$K = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(x+\xi)^2/4(t-\tau)}}{2\sqrt{t-\tau}} d\xi \quad (18)$$

Case Study 1/3

- To evaluate I, we define a new integration variable η such that $\eta = (x - \xi)/(2\sqrt{t - \tau})$.
- Also, we note that the exponent of the integrand for I vanishes at $\xi = x$, which is by definition within the domain, the interval of integration. So we divide the integration interval into $[0, x]$ and $[x, \infty]$ to express the solution in term of the error function.
- By the change of variable, we get

$$\begin{aligned} I &= \frac{1}{\sqrt{\pi}} \left[\int_{x/2\sqrt{t-\tau}}^{0} e^{-\eta^2} (-d\eta) + \int_{0}^{-\infty} e^{-\eta^2} (-d\eta) \right] \\ &= \frac{1}{\sqrt{\pi}} \left[\int_{0}^{x/2\sqrt{t-\tau}} e^{-\eta^2} \, d\eta + \int_{0}^{\infty} e^{-\eta^2} \, d\eta \right]. \end{aligned} \tag{19}$$

Case Study 1/3

- From the definition of the error function, we get

$$I = \frac{1}{2}\text{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) + \frac{1}{2}. \qquad (20)$$

- Evaluation of K is straightforward so we obtain

$$K = \frac{1}{\sqrt{\pi}}\left[\int_{x/2\sqrt{t-\tau}}^{0} e^{-\eta^2}\, d\eta\right] = \frac{1}{2}\text{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right). \qquad (21)$$

- With I and K, $w(x,t)$ is given as

$$w(x,t) = \int_0^t \dot{g}(\tau)\,\text{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right)\, d\tau + g(0)\,\text{erfc}\left(\frac{x}{2\sqrt{t}}\right) - g(t). \qquad (22)$$

- Since $u(x,t) = w(x,t) + g(t)$,

$$u(x,t) = \int_0^t \dot{g}(\tau)\,\text{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right)\, d\tau + g(0)\,\text{erfc}\left(\frac{x}{2\sqrt{t}}\right). \qquad (23)$$

Case Study 1/3

- In a special case $g(t) = c$ (constant), the half-space cooling solution is recovered:

$$u(x,t) = c\,\mathrm{erfc}(x/2\sqrt{t}).$$

- If $g(t) = c\cos(\omega t)$ representing a periodic heating,

$$u(x,t) = \int_0^t -c\omega\sin(\omega\tau)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t-\tau}}\right)d\tau + c\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right). \tag{24}$$

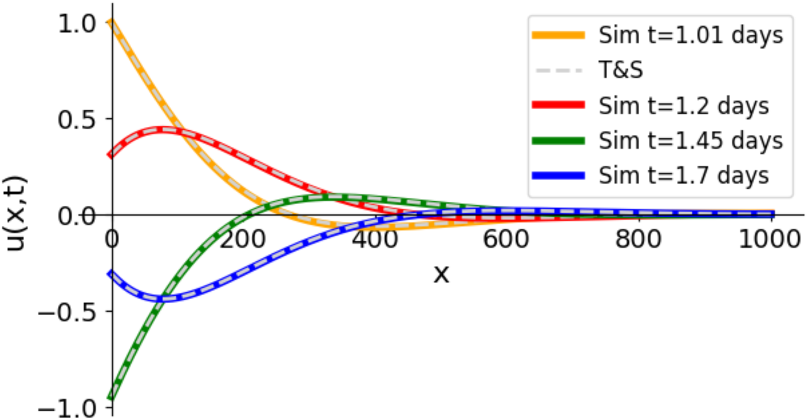
- We can get a different expression of $u(x,t)$ by integrating by parts the first term of (23):

$$u(x,t) = \frac{x}{2\sqrt{\pi}}\int_0^t \frac{g(\tau)e^{-x^2/4(t-\tau)}}{(t-\tau)^{3/2}}d\tau. \tag{25}$$

- The integration is not easy but we can always evaluate the solutions numerically. The tangible form of the solution is given in Sec. 4-14 of T&S.

Case Study 1/3

- Numerically evaluated similarity solutions show good agreement with the analytic solution given in Sec. 4-14[1]



[1] The two show good agreement for the tested values of t. However when $t \ll 1$ day or $t \gg 1$ day, they show significant discrepancy. I believe it suggests that we should be very careful when doing numerical integrations in (24) or (25).

Case Study 1/3

- However, it is more difficult to extract useful information directly from numerical solutions: e.g., surface heat flow.
- There might be a way of getting a closed form solution from (24) or (25).

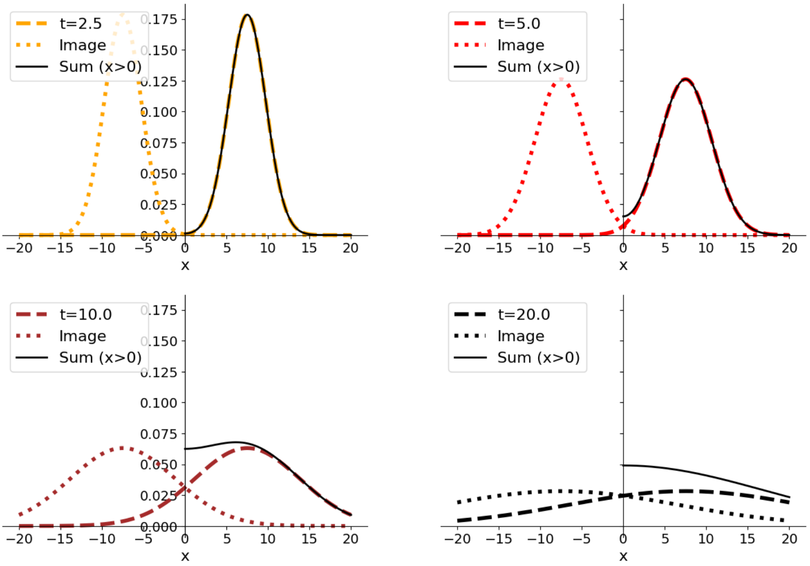
Case Study 2/3

- We also want to know the solution to the homogeneous Neumann type BVP.
- The purpose is to get the Green's function, which is the solution for the following equation:

$$u_t - u_{xx} = \delta(x-\xi)\delta(t),\ 0 \leq x < \infty,\ \xi > 0, \tag{26}$$
$$u_x(0,t) = 0,\ t > 0, \tag{27}$$
$$u(x,0) = 0. \tag{28}$$

- Like we obtained the Green's function for the homogeneous Dirichlet BVP, we use the image source technique. This time, however, we need a **positive** image source.

Case Study 2/3



Case Study 2/3

- So, our Green's function is the sum of two fundamental solutions:

$$G_N(x,\xi,t) \equiv F(x-\xi,t) + F(x+\xi,t). \tag{29}$$

- For the following homogeneous Neumann BVP,

$$u_t - u_{xx} = p(x,t),\ 0 \le x,\ 0 \le t, \tag{30}$$

$$u_x(0,t) = 0,\ \ t > 0, \tag{31}$$

$$u(x,0) = 0, \tag{32}$$

the solutions is

$$u(x,t) = \int_0^t d\tau \int_0^\infty p(\xi,\tau) G_N(x,\xi,t-\tau) d\xi. \tag{33}$$

Case Study 2/3

- If we have a non-zero initial condition, $u(x,0) = f(x)$, we can simply add the following contribution to the solution (33):

$$u(x,t) = \int_0^\infty f(\xi) G_N(x,\xi,t) d\xi. \tag{34}$$

- As in the inhomogeneous Dirichlet BVP, we can perform the homogenizing transformation for an inhomogeneous Neumann BVP with with $u_x(0,t) = h(t)$:

$$w(x,t) \equiv u(x,t) - x\, h(t). \tag{35}$$

- The solution boils down to this simplified form:

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_0^t h(\tau) \frac{e^{-x^2/4(t-\tau)}}{\sqrt{t-\tau}} d\tau. \tag{36}$$

Case Study 3/3

- The final case study is concerned about the Dirichlet BCs on a finite domain: $0 \leq x \leq L$.
- To enforce the homogeneous B.C. on the both ends of the domain, we need *infinite* number of image sources.
- Any finite sum will eventually fail to satisfy the boundary conditions. Let's try to understand this point by looking at a three-source example in the next slide.
- The Green's function for the heat conduction in a **finite domain with Dirichlet BCs** must be an infinite sum of the fundamental solutions:

$$G(x,\xi,t-\tau) \equiv \sum_{n=-\infty}^{\infty} \left[F(x-(2nL+\xi),t-\tau) - F(x-(2nL-\xi),t-\tau) \right]. \tag{37}$$

Case Study 3/3