Heat Advection-Diffusion

- We want to slightly generalize the heat diffusion equation to the heat **advection-diffusion** equation. The new equation will describe thermal energy that is not only diffused but also carried along with a deforming continuous medium.
- Note that the diffusion equation is derived from the conservation of thermal energy:

$$\frac{d}{dt}\int_{V(t)} \rho c_p T dV = \int_{\partial V(t)} k \nabla T \cdot \mathbf{n} dS. \tag{1}$$

- We previously assumed that the continuum body in which temperature is non-uniform such that diffusion occurs is not deforming. So, the volume V in the above equation was a constant.

Heat Advection-Diffusion

- Let's remove this assumption because when the medium is in motion, the volume is also time-dependent.
- By applying the Reynold's transport theorem to the l.h.s of Eq. (1) and the divergence theorem to the r.h.s, we get

$$\int_{v(t)} \left(\frac{\partial(\rho c_p T)}{\partial t} + \mathbf{v} \cdot \nabla(\rho c_p T) \right) dv = \int_{v(t)} \nabla \cdot (k \nabla T)\, dv. \tag{2}$$

- If the continuous media is *compressible*, deformation causes pV (pressure-volume, i.e., mechanical) work, which contributes the overall thermal energetics. However, if the media is **incompressible** or can **freely expand/contract**, it does not do any mechanical work.

Heat Advection-Diffusion

- Furthermore, when the continuous medium is going through *shearing*, in general we cannot ignore shear heating as a source term. In some cases, however, we can ignore shear heating. An example can be a plate with a prescribed thickness that is translating in one direction without internal deformation.
- If there are no other heat sources/sinks to consider, the assumptions of zero pV work, zero shear heating and constant material properties give

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T. \tag{3}$$

Compare it with Eq. (4-155) in T&S.

Mantle Adiabat I

- Mantle below the low-velocity zone and above the D'' layer is close to the *adiabatic or isentropic* state.
- This is a corollary from the prediction that vertical advection is the dominant heat transfer mechanism while others including conduction is negligible.
- Roughly speaking, the interior of the boundary layers are rather uniform in temperature and only the temperature increase as a result of compression by the weight of the overlying material matters.
- When quantified, this effect provide a prediction of mantle temperature distribution with depth or mantle geotherm.
- Since this mantle geotherm is based on the assumption of adiabaticity, it is also called an *adiabat*.
- Now, the meaning of *adiabatic temperature gradient* should be self-evident.

Mantle Adiabat II

- Some thermodynamic preliminary for quantifying the adiabatic mantle geotherm:
  - thermal expansion coefficient (α):

$$\alpha = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p, \tag{4}$$

    where ρ is the density, v is the specific volume ($1/\rho$), T is temperature and p is pressure.

  - Adiabatic compresibility (X_a) and adiabatic bulk modulus (K_a):

$$X_a \equiv -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s, \tag{5}$$

$$K_a \equiv \frac{1}{X_a}. \tag{6}$$

Mantle Adiabat III

- Specific heat at constant volume (c_v) and pressure (c_p):

$$c_v = \left(\frac{\delta q}{\delta T}\right)_v \tag{7}$$

$$c_p = \left(\frac{\delta q}{\delta T}\right)_p, \tag{8}$$

  where δq and δT are increments in thermal energy and temperature.

- From the definition of entropy, we get

$$dq = Tds. \tag{9}$$

  Using this, we also get

$$c_v = T\left(\frac{\delta s}{\delta T}\right)_v \tag{10}$$

$$c_p = T\left(\frac{\delta s}{\delta T}\right)_p. \tag{11}$$

Mantle Adiabat IV

- The heat energy balance equation can be written in terms of entropy:
$$T\frac{Ds}{Dt} = \frac{1}{Pe}\nabla^2 T + H^*, \tag{12}$$
where $Pe = wL/\kappa$ is the Péclet number, w is the vertical subduction speed, L is the slab length, κ is the thermal diffusivity and H^* is the dimensionless heat source stregnth, $HL/(\rho_m c_p \Delta T w)$.
- This form should be recognizable from the temperature form we know and (9).
- $Pe \sim 1500$ and $H^* \sim 0.01$. So, $Ds/Dt \sim 0$, justifying the isentropic assumption.

Mantle Adiabat V

- $$ds = \left(\frac{\partial s}{\partial T}\right)_p \frac{DT}{Dt} + \left(\frac{\partial s}{\partial p}\right)_T \frac{Dp}{Dt}, \tag{13}$$

  and from (11), one of the Maxwell relations

  $$\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p \tag{14}$$

  and (4), we get

  $$ds = \frac{c_p}{T} dT - \frac{\alpha_v}{\rho} dp. \tag{15}$$

Mantle Adiabat VI

- In the isentropic state, $ds = 0$. So, we get

$$\left(\frac{\partial T}{\partial p}\right)_s = \frac{\alpha_v T}{\rho c_p}. \tag{16}$$

Since $p = \rho g y$, the above equation becomes

$$\left(\frac{\partial T}{\partial y}\right)_s = \frac{\alpha_v g T}{c_p} = \frac{T}{H_T}, \tag{17}$$

where H_T is the adiabatic temperature scale height, $c_p/\alpha_v g$.

Mantle Adiabat VII

- α_v is difficult to measure for measure in the deep mantle. We use the Güneisen parameter (γ) to get a different form of the temperature gradient.

$$\gamma \equiv \frac{\alpha K_a}{\rho c_p} = \frac{\phi}{g H_T}, \tag{18}$$

  where ϕ is the seismic parameter defined as $\phi \equiv V_p^2 - 4/3 V_S^2 = K_a/\rho$.

- Finally, we get the adiabatic temperature gradient for mantle in terms of measurable quantities:

$$\left(\frac{\partial T}{\partial y}\right)_s = \frac{\gamma g T}{\phi}. \tag{19}$$

- g and ϕ are known rather precisely but γ is not.

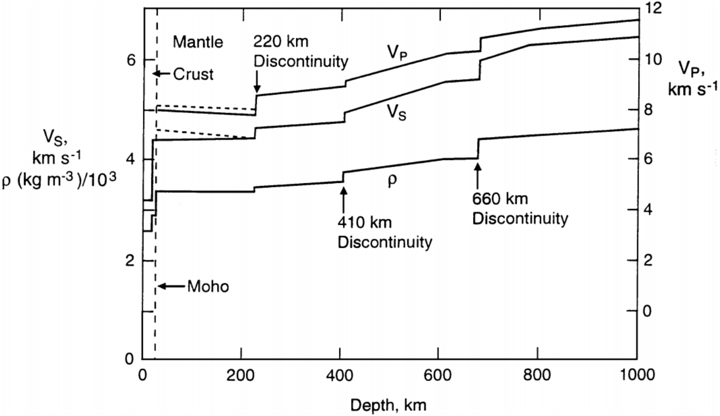
Mantle Adiabat VIII

- It used to be assumed that $\gamma\rho \sim$ constant but the following power-law formula is more commonly used:

$$\frac{\gamma}{\gamma_0} = \left(\frac{\rho_0}{\rho}\right)^n \tag{20}$$

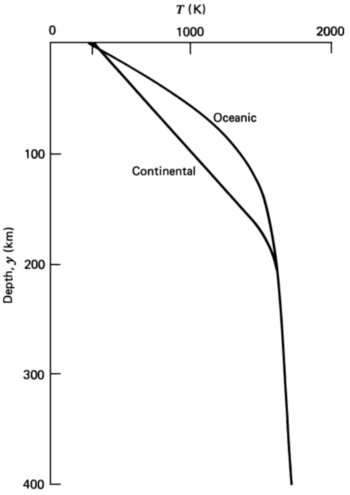
- γ_0, γ at the STP condition, for abundant upper mantle minerals are 1.2-1.8 and n is often assumed to be 1.
- For lower mantle, uncertainties are greater: $0.5 \leq n \leq 1$ and $1.2 \leq \gamma_0 \leq 1.5$.
- Density variations can be estimated from seismic data.
- In the end, the adiabatic gradient is estimated to be 0.35-045 K/km in the upper mantle and 0.25-0.4 K/km in the lower mantle.

Mantle Adiabat IX



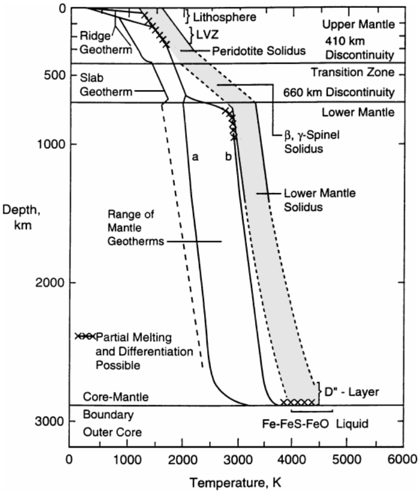
- Figure 4.55 Seismic velocities V_p and V_s and the density ρ are given as a function of depth.

Mantle Adiabat X



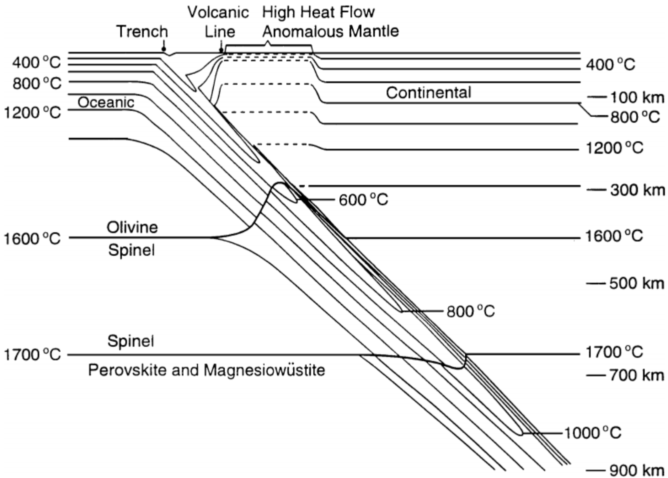
- Figure 4.56 Representative oceanic and continental shallow upper mantle geotherms.

Mantle Adiabat XI



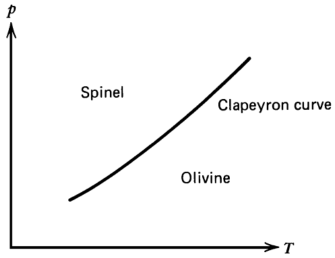
- Figure 4.57 Mantle geotherms are given for whole-mantle convection "(Curve *a*) and layered mantle convection" (Curve *b*). The range of values for the mantle solidus and the minimum temperatures in a subducted slab are also given.

Mantle Adiabat XII



- Figure 4.58 Isotherms (°C) in a typical descending lithosphere. The 410-km phase change is elevated in the subducted lithosphere. The position of the slip zone is also shown.

Mantle Adiabat XIII



- Figure 4.59 The Clapeyron or equilibrium curve separating two phases of the same material.

Mantle Adiabat XIV

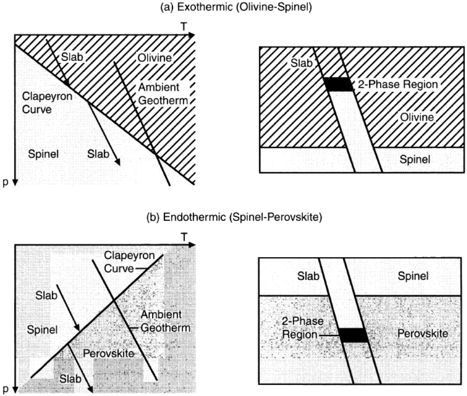
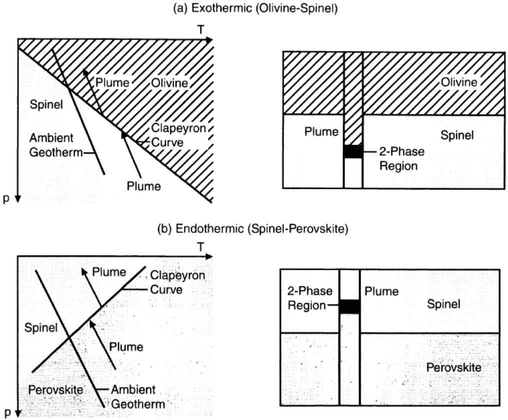


Figure 4.41. Sketch of equilibrium phase boundary displacement in a cold slab descending through (a) the exothermic olivine–spinel phase change and (b) the endothermic spinel–perovskite phase change. Motion through the phase transitions results in narrow two-phase regions in the slab. The positive Clapeyron slope of the exothermic phase change elevates the phase boundary in the slab while the negative Clapeyron slope of the endothermic phase change lowers the phase boundary in the slab. The p–T diagrams on the left show the path of the descending slab and the Clapeyron curves separating the phases. Univariant phase transitions are assumed.

From Mantle Convection in the Earth and Planets (2001, Cambridge University Press)

Mantle Adiabat XV



- Figure 4.42. Similar to Figure 4.41 but for a hot plume rising through the phase transitions. The olivine–spinel phase boundary is displaced downward in the plume while the spinel–perovskite phase boundary moves upward in the plume.

From Mantle Convection in the Earth and Planets (2001, Cambridge University Press)

Mantle Adiabat XVI

- Latent heat release or absorption during slab descent through the major phase changes also affects the thermal state of the slab. Coupled with thermal expansivity and phase boundary distortion, it affects the net body force on the slab as well. Latent heat is released during the slab's downward motion through the exothermic olivine–spinel phase change. The heat release tends to warm the slab. As a result of thermal expansion, the induced positive thermal anomaly exerts an upward body force on the slab, tending to retard its downward motion; the positive thermal anomaly also moves the phase boundary downward, contributing further to the body force resisting slab sinking. Latent heat release by the olivine–spinel phase transition leads to body forces opposing the downward motion of the descending slab.

  Latent heat is absorbed during the slab's downward motion through the endothermic spinel–perovskite phase change. The heat absorption tends to cool the slab. As a result of thermal contraction, the induced negative thermal anomaly exerts a downward body force on the slab, tending to assist its downward motion; the negative thermal anomaly also moves the phase boundary downward, providing an upward body force that resists slab sinking. Latent heat absorption by the spinel–perovskite phase transition both promotes and retards the downward motion of the descending slab.

p. 183 in Mantle Convection in the Earth and Planets (2001, Cambridge University Press)

Heat generation by radioactive heating and geotherms

- What is the reduced heat flux and why do we want to know it?
- How can we estimate *mantle* heat generation rate per mass (W/kg)?
- Why can't we assume a conduction geotherm for the mantle?
- What are the arguments for the exponential radioactivity distribution in the continental crust?

Periodic Heating of a Semi-Infinite Half-Space

- What is the skin depth?
- How deep should a bore hole be to avoid influences from diurnal, seasonal, or longer-term surface temperature variations?

Cooling of the Oceanic Lithosphere and the Plate Cooling Model

- When is the half-space cooling model a good approximation?
- What is the problem with the half-space cooling model?
- What is the problem with the plate model?

Stefan Problem and Solidification of a Dike or Sill

- What is the functional form of the dependence of solidification boundary (y_m) on time (t)?
- Can you dimensionalize $\theta = \frac{\mathrm{erf}(\eta)}{\mathrm{erf}(\lambda_1)}$ Eq. (4-137) and describe its behavior qualitatively?
- What's the difference between the solidification of a dike/sill and the Stefan problem?
- There is a typo in Eq. (4-140): $(T_m - T_0)/2\sqrt{\kappa T}$ should be $(T_m - T_0)/2\sqrt{\kappa t}$.

Ocean Floor Topography, Changes in Sea Level and Thermal and Subsidence History of Sedimentary Basins

- What is the principle applied to explaining the ocean floor topography (OFT)?
- How does the OFT depend on the age or equivalently the distance from a spreading center?
- What are the reasons for the $>$100 m sea level change over 10-100 Myrs? Can the time-dependence of radiogenic heat generation be responsible for it?
- Why is a basin shallower when a continental lithosphere is thinned as a whole than when only the continental crust is thinned?

Mantle Geotherm, Adiabats and Thermal Structure of the Subducted Lithosphere

- What is adiabat, adiabatic temperature gradient and adiabatic heating?
- How is the mantle geotherm shown in Fig. 4-57 constructed? In other words, can you explain why the mantle geotherm looks as shown?
- What is the origin of 410- and 660-km discontinuity? What does the corresponding Clapeyron slope imply for the buoyancy of a subducting slab?