Bender and Orszag, Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory, 1999, Springer-Verlag, N.Y.

the parabolic cylinder (Weber-Hermite) equation

$$y'' + (\nu + \tfrac{1}{2} - \tfrac{1}{4}x^2)y = 0, \tag{1.4.9}$$

and the Bessel equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0. \tag{1.4.10}$$

Some properties of the solutions to these and other classical differential equations are given in the Appendix.

1.5 INHOMOGENEOUS LINEAR EQUATIONS

Inhomogeneous linear differential equations are only slightly more complicated than homogeneous ones. This is because the difference of any two solutions of $Ly = f(x)$ is a solution of $Ly = 0$. As a result, the general solution of $Ly = f(x)$ is the sum of *any* particular solution of $Ly = f(x)$ and the general solution of $Ly = 0$.

Example 1 *General solution to an inhomogeneous equation.* Suppose $y = x$, $y = x^2$, and $y = x^3$ satisfy the second-order equation $Ly = f(x)$. Can you find the general solution without knowing the explicit form of L and f? The differences $x - x^2$ and $x^2 - x^3$ are both solutions of $Ly = 0$. These functions are linearly independent, so the general solution of $Ly = 0$ is $y(x) = c_1(x - x^2) + c_2(x^2 - x^3)$. Hence, the general solution of $Ly = f(x)$, which must contain two arbitrary constants of integration, is $y(x) = c_1(x - x^2) + c_2(x^2 - x^3) + x$.

All first-order linear inhomogeneous equations are soluble because it is always possible to find an integrating factor which is a function of x only. The integrating factor $I(x)$ for

$$y'(x) + p_0(x)y(x) = f(x) \tag{1.5.1}$$

is $I(x) = \exp\left[\int^x p_0(t)\, dt\right]$. Multiplying by $I(x)$ gives $I(x)y'(x) + p_0(x)y(x)I(x) = (d/dx)[I(x)y(x)] = f(x)I(x)$. So the solution of (1.5.1) is

$$y(x) = \frac{c_1}{I(x)} + \frac{1}{I(x)}\int^x f(t)I(t)\, dt. \tag{1.5.2}$$

Example 2 *First-order inhomogeneous equation.* The equation $y'(x) = y/(x + y)$ is not linear in y, but *is* linear in x! To demonstrate this, we simply exchange the dependent variable y with the independent variable x:

$$\frac{d}{dy}x(y) = \frac{x(y) + y}{y}.$$

An integrating factor for this equation is $I(y) = 1/y$. Multiplying by $I(y)$ gives $(d/dy) \times (x/y) = 1/y$ or $x(y) = y \ln y + c_1 y$.

The technique of exchanging the dependent and independent variables is essential for the solution of Prob. 1.22. A generalization of this method to partial differential equations is called the *hodograph transformation*.

There are several standard techniques for solving higher-order inhomogeneous linear equations.

Variation of Parameters

The only new complication in solving an inhomogeneous equation if the associated homogeneous equation is soluble is finding one particular solution. The method of *variation of parameters* is a general and infallible technique for determining a particular solution. The method could be classified as a super reduction of order.

We illustrate with a second-order equation. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous equation $Ly = 0$, where $L = d^2/dx^2 + p_1(x)\, d/dx + p_0(x)$. We seek a particular solution of $Ly = f(x)$ having the symmetric form

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x). \tag{1.5.3}$$

Of course, u_1 and u_2 are underdetermined so we have the freedom to impose a constraint which simplifies subsequent equations. We choose this constraint to be

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0. \tag{1.5.4}$$

Next, we differentiate (1.5.3) twice, substitute into $Ly = f(x)$, and remember that $Ly_1 = Ly_2 = 0$. Using (1.5.4) we have

$$u_1'(x)y_1'(x) + u_2'(x)y_2'(x) = f(x). \tag{1.5.5}$$

The solution of the simultaneous equations (1.5.4) and (1.5.5) for $u_1'(x)$ and $u_2'(x)$ is

$$u_1'(x) = -\frac{f(x)y_2(x)}{W(x)}, \qquad u_2'(x) = \frac{f(x)y_1(x)}{W(x)}, \tag{1.5.6}$$

where $W(x) = W[y_1(x), y_2(x)]$ is the Wronskian. Observe that the denominators W do not vanish because $y_1(x)$ and $y_2(x)$ are assumed to be linearly independent solutions of $Ly = 0$.

Integrating (1.5.6) gives the final expression for the particular solution in (1.5.3):

$$y(x) = -y_1(x)\int^x \frac{f(t)y_2(t)}{W(t)}\, dt + y_2(x)\int^x \frac{f(t)y_1(t)}{W(t)}\, dt. \tag{1.5.7}$$

Example 3 *Variation of parameters.* To solve $y'' - 3y' + 2y = e^{4x}$ by variation of parameters, we first determine that two solutions of the associated homogeneous equation are $y_1 = e^x$ and $y_2 = e^{2x}$. Next we compute the Wronskian: $W(e^x, e^{2x}) = e^{3x}$. Substituting into (1.5.7) gives

$$\begin{aligned} y(x) &= -e^x \int^x dt\, e^{4t}e^{2t}e^{-3t} + e^{2x}\int^x dt\, e^{4t}e^{t}e^{-3t} \\ &= c_1 e^x + c_2 e^{2x} + \tfrac{1}{6}e^{4x}, \end{aligned}$$

which is the general solution to the inhomogeneous differential equation.

Variation of parameters for nth-order equations is discussed in Prob. 1.15.

Green's Functions

There is another general method for constructing the solution to an inhomogeneous linear differential equation which is equivalent to variation of parameters. This method represents the solution as an integral over a *Green's function*.

To define a Green's function it is necessary to introduce the Dirac delta function $\delta(x-a)$. This function may be thought of as a mathematical idealization of a unit impulse; it is an infinitely thin spike centered at $x = a$ having unit area.† The δ function has two defining properties. First,

$$\delta(x-a) = 0, \qquad x \neq a. \tag{1.5.8a}$$

Second,

$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1. \tag{1.5.8b}$$

From these properties we have the crucial result (see Prob. 1.16) that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a) \tag{1.5.9}$$

if $f(x)$ is continuous at a.

There are many ways to represent the δ function. It may be expressed (nonuniquely) as the limit of a sequence of functions:

$$\delta(x-a) = \lim_{\varepsilon\to 0+} F_\varepsilon(x), \tag{1.5.10a}$$

where

$$F_\varepsilon(x) = \begin{cases} 0, & x < a - \tfrac{1}{2}\varepsilon, \\ 1/\varepsilon, & a - \tfrac{1}{2}\varepsilon \le x \le a + \tfrac{1}{2}\varepsilon, \\ 0, & a + \tfrac{1}{2}\varepsilon < x; \end{cases}$$

or

$$\delta(x-a) \equiv \lim_{\varepsilon\to 0+} \frac{\varepsilon}{\pi[(x-a)^2 + \varepsilon^2]}; \tag{1.5.10b}$$

or

$$\delta(x-a) \equiv \lim_{\varepsilon\to 0+} (\pi\varepsilon)^{-1/2} e^{-(x-a)^2/\varepsilon}; \tag{1.5.10c}$$

or

$$\delta(x-a) \equiv \lim_{L\to +\infty} \frac{1}{2\pi} \int_{-L}^{L} e^{i(x-a)t}\, dt. \tag{1.5.10d}$$

(The notation $\varepsilon \to 0+$ means that ε approaches 0 through positive values only.) It is easy to verify (see Prob. 1.17) that the formulations in (1.5.10) satisfy (1.5.8).

Alternatively, $\delta(x-a)$ may be viewed as the derivative of a discontinuous function. If $h(x-a)$ is the Heaviside step function defined by

$$h(x-a) \equiv \begin{cases} 0, & x < a, \\ \tfrac{1}{2}, & x = a, \\ 1, & x > a, \end{cases}$$

then (see Prob. 1.18)

$$\delta(x-a) \equiv \frac{d}{dx} h(x-a). \tag{1.5.11}$$

† Technically, the δ function is not really a function; it is a distribution (see References).

Notice that integration is a *smoothing* operation but that differentiation is unsmoothing. For example, the Heaviside function, which is the integral of the δ function,

$$h(x-a) = \int_{-\infty}^{x} \delta(t-a)\, dt,$$

has a finite jump discontinuity while the δ function has an infinite jump discontinuity. Similarly, the ramp function, which is the integral of the Heaviside function,

$$r(x-a) = \int_{-\infty}^{x} h(t-a)\, dt = \begin{cases} 0, & x \le a, \\ x-a, & a \le x, \end{cases}$$

is continuous everywhere.

Next, we define the Green's function. The Green's function $G(x, a)$ associated with the inhomogeneous equation $Ly = f(x)$ satisfies the differential equation

$$LG(x, a) = \delta(x-a). \tag{1.5.12}$$

Once $G(x, a)$ is known, it is easy to represent the solution to $Ly = f(x)$ as an integral

$$y(x) = \int_{-\infty}^{\infty} da\, f(a) G(x, a). \tag{1.5.13}$$

To verify that $y(x)$ in (1.5.13) solves $Ly = f$, we differentiate under the integral:

$$\begin{aligned} Ly(x) &= L \int_{-\infty}^{\infty} da\, f(a) G(x, a) \\ &= \int_{-\infty}^{\infty} da\, f(a) LG(x, a) \\ &= \int_{-\infty}^{\infty} da\, f(a) \delta(x-a) \\ &= f(x), \end{aligned}$$

where we have used (1.5.12) and (1.5.9) in turn.

The only remaining problem is to solve (1.5.12) for $G(x, a)$. But this is easy once the solutions to the associated homogeneous equation $Ly = 0$ are known. To illustrate we solve the second-order equation

$$LG(x, a) = \left[\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_0(x)\right] G(x, a) = \delta(x-a). \tag{1.5.14}$$

We denote two linearly independent solutions to $Ly = 0$ by $y_1(x)$ and $y_2(x)$. Then, when $x \neq a$, the right side of (1.5.14) vanishes and we have

$$G(x, a) = A_1 y_1(x) + A_2 y_2(x), \qquad x < a,$$
$$G(x, a) = B_1 y_1(x) + B_2 y_2(x), \qquad x > a.$$

In order to relate the solution for $G(x, a)$ for $x < a$ to the solution for $x > a$, we argue that $G(x, a)$ is continuous at $x = a$ and that $\partial G/\partial x$ has a finite jump discontinuity of magnitude 1 at $x = a$. To show this, we observe that the most singular term on the left side of the Green's function equation (1.5.14) must be $\partial^2 G/\partial x^2$ because differentiation is an unsmoothing operation; if G or $\partial G/\partial x$ had an infinite jump discontinuity at $x = a$ like that of a δ function, then $\partial^2 G/\partial x^2$ would be even more singular than a δ function and (1.5.14) could not be satisfied. Thus, (1.5.14) implies that $\partial^2 G/\partial x^2 - \delta(x - a)$ must be less singular than a δ function at $x = a$. Therefore, integrating $\partial^2 G/\partial x^2 - \delta(x - a)$ from $-\infty$ to x gives a function which is continuous even at $x = a$: $\partial G/\partial x - h(x - a)$ is continuous everywhere. Hence the discontinuity in $\partial G/\partial x$ at $x = a$ is the same as that of the Heaviside function $h(x - a)$:

$$\lim_{\varepsilon \to 0+} \left[\frac{\partial G}{\partial x}\bigg|_{x=a+\varepsilon} - \frac{\partial G}{\partial x}\bigg|_{x=a-\varepsilon} \right] = 1. \tag{1.5.15}$$

Finally, since $\partial G/\partial x$ has only a finite jump discontinuity, its indefinite integral $G(x, a)$ must be continuous at $x = a$.

Continuity of $G(x, a)$ at $x = a$ gives the condition

$$A_1 y_1(a) + A_2 y_2(a) = B_1 y_1(a) + B_2 y_2(a).$$

Also, (1.5.15) requires that

$$B_1 y_1'(a) + B_2 y_2'(a) - A_1 y_1'(a) - A_2 y_2'(a) = 1.$$

Using these relations and solving for $B_1 - A_1$ and $B_2 - A_2$, we obtain

$$B_1 - A_1 = -\frac{y_2(a)}{W[y_1(a), y_2(a)]}, \tag{1.5.16}$$

$$B_2 - A_2 = \frac{y_1(a)}{W[y_1(a), y_2(a)]}. \tag{1.5.17}$$

Observe the strong parallel between these equations and (1.5.6).

We have now completed the solution of the Green's function equation (1.5.14). However, A_1 and A_2 are still arbitrary because $G(x, a)$ is only determined by (1.5.14) up to a solution of the homogeneous equation. Choosing $A_1 = A_2 = 0$ and using (1.5.16) and (1.5.17) to determine B_1 and B_2, we obtain

$$G(x, a) = \begin{cases} \dfrac{-y_2(a)y_1(x) + y_1(a)y_2(x)}{W[y_1(a), y_2(a)]}, & x \ge a, \\ 0, & x < a. \end{cases} \tag{1.5.18}$$

Substituting this formula for $G(x, a)$ into (1.5.13) reproduces exactly the variation of parameters result in (1.5.7).

The Green's function approach has a distinct advantage over the method of variation of parameters when it is necessary to solve a differential equation $Ly = f$ where L and the boundary conditions are fixed but f ranges over a wide variety of functions. (Why?) The analysis is particularly simple when the boundary conditions are homogeneous.

Example 4 *Solution of a boundary-value problem by Green's functions.* The Green's function for the boundary-value problem $y'' = f(x)[y(0) = 0, y'(1) = 0]$ is defined by the equations

$$(\partial^2 G/\partial x^2)(x, a) = \delta(x - a), \qquad G(0, a) = 0, \qquad (\partial G/\partial x)(1, a) = 0.$$

Notice that we have chosen G to satisfy the same *homogeneous* boundary conditions as y. The solution for $G(x, a)$ is

$$G(x, a) = \begin{cases} -x, & x < a, \\ -a, & x \ge a, \end{cases}$$

when $0 < a < 1$. For any $f(x)$, $y(x)$ can then be represented as $y(x) = \int_0^1 G(x, a) f(a)\, da$ $(0 \le x \le 1)$. Note that we do not integrate from $-\infty$ to $+\infty$. Why?

Example 5 *Solution of a boundary-value problem by Green's functions.* The Green's function for the boundary-value problem $y'' - y = f(x)[y(\pm\infty) = 0]$ is defined by the equations $\partial^2 G/\partial x^2 - G(x, a) = \delta(x - a)$, $G(\pm\infty, a) = 0$. The solution for $G(x, a)$ is $G(x, a) = -\frac{1}{2}e^{-|x-a|}$. Thus, for any $f(x)$, $y(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-a|} f(a)\, da$.

Reduction of Order

For the sake of completeness, it is important to state that *reduction of order* reduces the order of inhomogeneous as well as homogeneous equations. Thus, since all first-order linear equations are soluble, reduction of order is especially useful for second-order linear equations.

Example 6 *Reduction of order for an inhomogeneous equation.* One solution of the homogeneous equation $a(x)y'' + xy' - y = 0$ is $y_1(x) = x$. Therefore, to solve the inhomogeneous equation $a(x)y'' + xy' - y = f(x)$ by reduction of order, we seek a solution of the form $y(x) = y_1(x)u(x) = xu(x)$. Substituting gives a first-order equation for $u'(x)$ which is easy to solve: $xa(x)u'' + [2a(x) + x^2]u' = f(x)$.

Method of Undetermined Coefficients

There is another technique for determining a particular solution to $Ly = f(x)$ called the method of *undetermined coefficients*, which we discuss briefly. This method is really little more than organized guesswork, but when it works it is faster than variation of parameters. Its application is usually limited to constant-coefficient equations where $f(x)$ is an additive or multiplicative combination of e^x, $\sin x$, $\cos x$, and polynomials in x, or equidimensional equations where $f(x)$ is a polynomial in x.

Example 7 *Method of undetermined coefficients.*

(*a*) To solve $y''' + y = e^x \sin x$ we guess a particular solution of the form $y = ae^x \sin x + be^x \cos x$ and determine the "undetermined coefficients" a and b by substituting into the differential equation. The results are $a = -\frac{1}{5}$ and $b = -\frac{2}{5}$.