Rayleigh-Taylor Instability

- Experiments with saline water
  `http://youtu.be/NI85oC-3mJ0`
- Stability by surface tension
  `https://youtu.be/yutbmcO5g2o?si=aEZXm8LCBB5LSPxv&t=1076`
  (17 m 53 s - 19 m 48 s)

Convection analysis: Three-part approach

- Stability analysis
  - We'd like to tell whether a fluid layer is going to start convecting or not.
  - We will introduce an analytical technique called *linear stability analysis*.
  - Sec. 6. 19 in T&S (3rd ed.)
- Transient growth of boundary layers. Sec. 6.20.
- Steady-state boundary-layer theory. Sec. 6.21.

Stability Analysis: Basic Concepts I

- We will learn **on what ground the mantle convection was first proposed and considered to be plausible**.
- In spite of the complexity of the governing equations describing the conservation of mass, momentum and energy, sometimes very simple flow patterns appear (e.g., laminar channel flows).
- These patterns, however, can be realized only for certain ranges of the parameters characterizing them (e.g., transition to a turbulent flow as Re increases.)
- Outside these ranges, simple flow patterns cannot be realized because of their **inherent instability**, i.e. in their inability to sustain themselves against small perturbations to which any physical system is subject.

Stability Analysis: Basic Concepts II

- The problems of **hydrodynamic stability** are concerned about the transition from the stable to the unstable patterns of permissible flows.
- We will closely follow Chandrasekha's treatise,"Hydrodynamic and Hydromagnetic Stability" (Oxford University Press, 1961 or Dover, 1981).
- The key question addressing the concept of stability: **If a system is disturbed, will the disturbance gradually die down, or will the disturbance grow in amplitude in such a way that the system progressively departs from the initial state and never reverts to it?**
- In the former case, we say the system is **stable** w.r.t. the particular disturbance; and in the latter case, we say the system is **unstable.**
  - Watch 0m 30s-1m 55s of `https://www.youtube.com/watch?v=xe-f4gokRBs`

Stability Analysis: Basic Concepts III

- Watch 0-5m 15s, 19m 49s-22m 50s: `https://www.youtube.com/watch?v=yutbmc05g2o&index=19&list=PL0EC6527BE871ABA3`
- Kelvin-Helmholtz instability: `https://www.youtube.com/watch?v=UbAfvcaYr00`

- In the space of parameters, all initial conditions can be classified as either stable or unstable according to the criterion stated above. Then, it will be possible to find in the same parameter space the locus that separates the two states.
- The locus defines the states of **marginal stability** of the system. By this definition, a marginal state is a state of **neutral stability** (i.e., neither stable nor unstable).
- Finding marginal states is the prime objective of the hydrodynamic stability analysis[1].

Stability Analysis: Basic Concepts IV

- It is convenient to assume all the parameters are fixed except one, which is continuously varied. When the particular parameter being varied takes a certain critical value, the system will pass from a stable to an unstable state.
- We say that **instability sets in** at this value of the chosen parameter.
- The marginal stability can be further divided according to the growth or decay of a given perturbation is *monotonic* or *oscillating*. The latter case is called "overstability". For further details, read Sec. 2 of Chandrasekhar's book.

[1] The term "stability analysis" is also used in the study of a system of ODEs. There is some conceptual similarity but they greatly differ in practice.

Stability Analysis: Procedure

The general procedure of a stability problem is as follows:

1. Assume an initial flow representing a stationary state of the system.
2. **Linearize** the governing equation. In other words, rewrite equations in terms of **infinitesimal perturbations** to the physical variables. This step is called “linearization” because we neglect all the products and powers (higher than the first) of the increments and **retain only the terms which are linear** in the increments.
3. The solution you get from the linearized equations will tell you the perturbation would grow or get damped.

Stability Analysis: Procedure cont'd

- Since the stability is determined with respect to all the possible perturbations, we need a **complete representation** of them.
- For instance, if we represent the perturbation with an amplitude function, A, a Fourier series or spherical harmonic representation of A would be appropriate:

$$A(\mathbf{r}, t) = \int A_{\mathbf{k}}(\mathbf{r}, t) d\mathbf{k}, \tag{1}$$

  where $\mathbf{r}$ is a position vector (e.g., (x, y, z)), $\mathbf{k}$ is a wavenumber vector (e.g., (k_x, k_y, k_z))
- We then separate the dependence on time by seeking solutions of the form

$$A_{\mathbf{k}}(\mathbf{r}, t) = A_{\mathbf{k}}(\mathbf{r}) e^{p_{\mathbf{k}} t}, \tag{2}$$

  where $p_{\mathbf{k}}$ is a constant to be determined.

Stability Analysis: Procedure cont'd

- In general, the characteristic values for $p_{\mathbf{k}}$ will be complex:

$$p_{\mathbf{k}} = \xi_{\mathbf{k}} + i\zeta_{\mathbf{k}} \tag{3}$$

- The imaginary part will correspond to the oscillatory change in the amplitude of perturbations while the sign of real part will determine whether the perturbation will grow or decay.
- As we've seen in many previous examples (e.g., Sec. 6.12 Diapirism and Sec. 6.13 Folding in T&S), the growth rate, $p_{\mathbf{k}}$, is **a combination of parameters** (cf. viscous relaxation time, Reynolds number, etc).
- Therefore, the marginal stability is defined as the envelope of loci $p_{\mathbf{k}}(X_1, X_2, \ldots) = 0$, where X_i is the i-th parameter.

Dimensionless Numbers

- Non-dimensionalizing the governing equations typically yields non-dimensional numbers and their physical meanings become clear during the process.
- **Reynolds Number.** Non-dimensionalize the Navier-Stokes equation for incompressible, constant-viscosity Newtonian fluid in terms of the following non-dimensional variables:

$$\mathbf{v}' = \mathbf{v}/V, \quad p' = p/(\rho V^2), \quad \mathbf{f}' = \mathbf{f}\,D/(\rho V^2),$$
$$\partial/\partial t' = (D/V)\partial/\partial t, \quad \nabla' = D\nabla,$$

where D and V are the characteristic length and speed. Then, we get

$$\begin{aligned} \frac{D\mathbf{v}'}{Dt'} &= -\nabla' p' + \frac{\mu}{\rho D V}\nabla'^2\mathbf{v}' + \mathbf{f}' \\ &= -\nabla' p' + \frac{1}{Re}\nabla'^2\mathbf{v}' + \mathbf{f}' \end{aligned} \tag{4}$$

Dimensionless Numbers

- **Rayleigh Number (Ra).** Non-dimensionalize the Stokes equation,

$$0 = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho g \mathbf{e}_g, \tag{5}$$

in terms of the following non-dimensional variables[2]:

$$\begin{aligned} p' &= p/(\kappa\mu/D^2), \\ \mathbf{v}' &= \mathbf{v}/(\kappa/D), \\ \rho' &= \rho/\rho_0, \quad g' = g/g_0, \quad \mu' = \mu/\mu_0. \end{aligned}$$

By dividing the equation by the common factor from the first two terms, $\kappa\mu/D^3$, we get

$$\begin{aligned} 0 &= -\nabla' p' + \mu' \nabla'^2 \mathbf{v}' + \frac{\rho_0 g_0 D^3}{\kappa\mu} \rho' g' \mathbf{e}_g \\ &= -\nabla' p' + \mu' \nabla'^2 \mathbf{v}' + \mathrm{Ra}\, \rho' g' \mathbf{e}_g \end{aligned} \tag{6}$$

[2]Unnlike in the derivation of the Reynolds number, all the time dimension is provided by thermal diffusivity, κ [m^2/s]

Dimensionless Numbers

- **Prandtl Number (Pr).** Non-dimensionalize the non-steady Stokes equation

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho g \mathbf{e}_g, \tag{7}$$

in terms of the previous non-dimensional variables.
As before, dividing the equation by $\kappa\mu/D^3$, we get

$$\begin{aligned} -\nabla' p' + \mu' \nabla'^2 \mathbf{v}' + \mathrm{Ra}\, \rho' g' \mathbf{e}_g &= \rho_0 \frac{\kappa}{D} \frac{\kappa}{D^2} \frac{D^3}{\kappa\mu} \rho' \frac{D'\mathbf{v}'}{D't'} \\ &= \frac{\kappa}{\mu/\rho_0} \rho' \frac{D'\mathbf{v}'}{D't'} = \frac{\kappa}{\nu} \rho' \frac{D'\mathbf{v}'}{D't'} = \frac{1}{\mathrm{Pr}} \rho' \frac{D'\mathbf{v}'}{D't'}. \end{aligned} \tag{8}$$

Pr = ν/κ and when Pr is infinite, i.e., ν is big and κ is small, we can ignore the inertial term as a whole and the Stokes equation (5) is retrieved.

Dimensionless Numbers

- **Péclet Number (Pe).** Non-dimensionalize the heat advection-diffusion euqation,

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T, \tag{9}$$

in terms of the following non-dimensional variables:
  - $T' = T/T_0, \quad \mathbf{v}' = \mathbf{v}/V, \quad \kappa' = \kappa/\kappa_0,$
  - $\nabla' = D\nabla, \quad \partial/\partial t' = (D^2/\kappa_0)\partial/\partial t.$

Dividing the equation by $T_0\kappa_0/D^2$, we get

$$\frac{\partial T'}{\partial t'} + \frac{VT_0}{D}\frac{D^2}{T_0\kappa_0}\mathbf{v}' \cdot \nabla' T' = \kappa' \nabla'^2 T',$$

where the coefficient of the advection term becomes Pe = VD/κ_0. So, the non-dimensional heat equation becomes

$$\frac{\partial T'}{\partial t'} + \mathrm{Pe}\, \mathbf{v}' \cdot \nabla' T' = \kappa' \nabla'^2 T'. \tag{10}$$

- Try to infer the physical meaning of Pe from this equation.

Dimensionless Numbers

- **Nussel Number (Nu)** is defined as

$$Nu \equiv \frac{\text{Convective heat flux}}{\text{Conductive heat flux}}. \tag{11}$$

- Apparently, there is no corresponding non-dimensional governing equation.
- However, the physical meaning is clear from the definition itself.

Thermal Convection: Governing Equations

- We have studied heat transfer by diffusion and fluid motion separately or together only in the pipe flow case.
- Convection is a way of transferring heat energy in which the amount of heat carried by a moving medium is more significant than by diffusion. So, to study convection, we need to consider both less restricted fluid motions and conductive heat transfer.
- The governing equations are the Stokes equation (5)

$$0 = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho g \mathbf{e}_g,$$

and the heat advection-diffusion equation (9)

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T.$$

for no heat sources/sinks, zero pV work, zero shear heating and constant material properties.

Thermal Convection: Boussinesq Approximation

- Boussinesq approximaton: Density change due to thermal expansion is assumed to contribute only to the buoyancy force.

$$\rho = \rho_0 + \rho' = \rho_0 - \alpha_v \rho_0 (T - T_0), \qquad (12)$$

  where ρ_0 is the reference density at $T = T_0$.
- All the other terms in (5) and (9) are free from the effects of density change.
- Beyond the linear stability analysis, temperature dependence of viscosity should be considered and it requires a numerical approach to solve the governing equations.
- The Boussinesq approximation might be too simple for the Earth's mantle. Research on this **compressible mantle convection** has started only recently.

Thermal Convection: Linearization

- Let's consider an infinite horizontal layer of fluid of thickness b (see Fig. 6-38 of T&S). The bottom temperature (T_1) is higher than the top temperature (T_0).
- Further assume that initially, there is no motions

$$\mathbf{v} = \mathbf{0}, \tag{13}$$

and the temperature distribution is in a steady state. Then, Eq. (9) becomes

$$\nabla^2 T = \frac{\partial^2 T}{\partial y^2} = 0, \tag{14}$$

From the boundary conditions assumed earlier, the temperature profile becomes linear:

$$T_c = \frac{T_1 + T_0}{2} + \frac{T_1 - T_0}{b} y, \tag{15}$$

where the subscript c means that this is a conductive profile.

Thermal Convection: Linearization

- As for the momentum equation, only the y component is non-trivial:

$$0 = -\frac{\partial p}{\partial y} + \rho g = -\frac{\partial p}{\partial y} + \rho_0 g(1 - \alpha_v(T_c(y) - T_0)), \quad (16)$$

  which can be integrated to get an expression for p, which is denoted as p_c because it is associated with T_c.

- Equations (13), (15) and (16) completely describes the initial state.
- Now, let the initial state be slightly perturbed such that

$$T' = T - T_c, \quad (17)$$

  where T is the perturbed temperature field and T' is the added small perturbation.

- This perturbation will generate velocity field $\mathbf{v}' = (u'(x, y), v'(x, y))$ and dynamic pressure $P'(x, y)$.

Thermal Convection: Linearization

- The perturbed variables, $\mathbf{v} = \mathbf{0} + \mathbf{v}'$, $p = p_c + P'$ and $T = T_c + T'$, should still satisfy the governing equations.
- Heat equation:

$$\frac{\partial (T_c + T')}{\partial t} + \mathbf{v}' \cdot \nabla (T_c + T') = \kappa \nabla^2 (T_c + T'). \tag{18}$$

  Discarding any term that is quadratic or higher-order in terms of the perturbed variables, we get

$$\frac{\partial T'}{\partial t} + v' \frac{T_1 - T_0}{b} = \kappa \nabla^2 T', \tag{19}$$

  where v' is the y component of $\mathbf{v}'$. This is equal to Eq. 6-302 of T&S.

Thermal Convection: Linearization

- Momentum equation:

$$0 = -\frac{\partial(p_c + P')}{\partial x} + \mu\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right),$$

$$0 = -\frac{\partial(p_c + P')}{\partial y} + \mu\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right) + \rho_0 g(1 - \alpha_v((T_c(y) + T') - T_0))$$

From the fact that p_c is a function of y only and from Eq. (16),

$$0 = -\frac{\partial P'}{\partial x} + \mu\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right), \tag{20}$$

$$0 = -\frac{\partial P'}{\partial y} - \rho_0 g \alpha_v T' + \mu\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right). \tag{21}$$

Thermal Convection: Linearization

- Continuity equation:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0. \tag{22}$$

- Equations (19), (20), (21) and (22) completes the set of equations that describe the disturbed motion of the fluid. They are said to be **linearized**.

Thermal Convection: Boundary Conditions

- Impermeable and isothermal conditions:

$$T' = v' = 0 \quad \text{on} \quad y = \pm\frac{b}{2}. \tag{23}$$

- Possible conditions for horizontal velocity:
  - No-slip condition

$$u = 0 \quad \text{on} \quad y = \pm\frac{b}{2}. \tag{24}$$

  - Free-slip condition ($\equiv$ zero horizontal traction)

$$\tau_{xy} = \mu\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) = 0 \quad \text{on} \quad y = \pm\frac{b}{2}. \tag{25}$$

    where τ_{xy} is a component of deviatoric stress.
    This condition allows a simpler solution and is further simplified to

$$\frac{\partial u'}{\partial y} = 0 \quad \text{on} \quad y = \pm\frac{b}{2}, \tag{26}$$

    since $v' = 0$ for any x and thus $\partial v'/\partial x = 0$.

Thermal Convection: Stability Analysis

- We can consolidate the momentum equations into a biharmonic equation of a single scalar variable using the stream function:

$$0 = \mu \left(\frac{\partial^4 \psi'}{\partial x^4} + 2\frac{\partial^4 \psi'}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi'}{\partial y^4} \right) - \rho_0 g \alpha_v \frac{\partial T'}{\partial x}. \qquad (27)$$

  Here, it should be obvious that the linearized equations of the velocity perturbations becomes the linearized equation of the perturbed stream function.

- The perturbations are assumed to take the following forms:

$$\psi' = \psi_0' \cos\frac{\pi y}{b} \sin\left(\frac{2\pi x}{\lambda}\right) e^{\alpha' t}, \qquad (28)$$

$$T' = T_0' \cos\frac{\pi y}{b} \sin\left(\frac{2\pi x}{\lambda}\right) e^{\alpha' t}. \qquad (29)$$

  Note that these forms automatically satisfy the boundary[3] conditions.

[3] no-slip or free-slip?

Thermal Convection: Stability Analysis

- Plugging in (28) and (29) into the governing equations, we get

$$\left(\alpha' + \frac{\kappa\pi^2}{b^2} + \frac{\kappa 4\pi^2}{\lambda^2}\right) T_0' = -\frac{(T_1 - T_0)2\pi}{\lambda b}\psi_0', \qquad (30)$$

$$\mu\left(\frac{4\pi^2}{\lambda^2} + \frac{\pi^2}{b^2}\right)^2 \psi_0' = -\frac{2\pi}{\lambda}\rho_0 g \alpha_v T_0'. \qquad (31)$$

- Solving the above system of equation for the **growth rate**, α',

$$\alpha' = \frac{\kappa}{b^2}\left[Ra\frac{k^2}{(k^2+\pi^2)^2} - (\pi^2 + k^2)\right], \qquad (32)$$

where the **Rayleigh number** Ra is defined as

$$Ra = \frac{\rho_0 g \alpha_v (T_1 - T_0) b^3}{\mu\kappa} \qquad (33)$$

and k is a dimensionless wave number, $2\pi b/\lambda$.

Thermal Convection: Stability Analysis

- The marginal (or neutral) state, in which the perturbation does not grow nor decay spontaneously, corresponds to $\alpha' = 0$. In this case,

$$Ra = \frac{(k^2 + \pi^2)^3}{k^2}, \tag{34}$$

  and this value is called the **critical Rayleigh number,** Ra_{cr}.
- If $Ra > Ra_{cr}$, $\alpha' > 0$, which means the perturbation will exponentially grow in time; if $Ra < Ra_{cr}$, $\alpha' < 0$ and the perturbation will die down.
- As we discussed earlier in a general setting, knowing the marginal state is sufficient for understanding the stability of a flow system. That's why Ra_{cr} as well as Ra are important for convection.

Thermal Convection: Stability Analysis

- Fig. 6-39 of T&S is the $Ra_{cr} - k$ plot.
- The smallest possible Ra_{cr} and the corresponding k can be found from $\partial Ra_{cr}/\partial k = 0$ and they are 657.5 and $\pi/\sqrt{2}$, respectively.
- Since $k = 2\pi b/\lambda$, the wavelength λ is $2\sqrt{2}b$.
- For the no-slip condition, $Ra_{cr} = 1707.8$ and $\lambda = 2.016b$.
- A situation closer to the Earth mantle: Cooled from the top, internal heating, no heat flux through the bottom

$$Ra_H = \frac{\alpha_V \rho^2 g H b^2}{k \mu \kappa}, \qquad (35)$$

  where k is the heat conductivity, not wave number!
  - No-slip conditions: $Ra_{cr} = 2772$ and $\lambda = (2\pi/2.63)b$.
  - Free-slip condition $Ra_{cr} = 867.8$ and $\lambda = (2\pi/1.79)b$.

Thermal Convection: Stability Analysis

- For the Earth, we take
  - $\mu = 10^{21}$ Pa·s,
  - $k = 4$ W/m/K,
  - $\kappa = 10^{-6}$ m^2/s,
  - $\alpha_v = 3\times10^{-5}$ /K
  - g = 10 m/s^2,
  - $\rho_0 = 4000$ kg/m^3,
  - $H = 9\times10^{-12}$ W/kg.

  Then for the upper mantle convection, $b = 700$ km and $Ra_H = 2 \times 10^6$; for the whole mantle convection, $b = 2880$ km and $Ra_H = 2 \times 10^9$.
- In either case, the Earth mantle is supposed to be unstable. Based on this, Arthus Holmes argued in 1931 that the thermal convection in the mantle should be vigorous enough to drive continental drift.

Convection Initiation in Everyday Materials I

Properties	Water	Vegetable oil	Corn syrup
k (W/m/K)	0.6	0.2	0.3
c_p (J/Kg/K)	4200	2200	2700
ρ (kg/m^3)	1000	900	1400
κ (m^2/s)			
α (K^{-1})	2×10^{-4}	7×10^{-4}	5.6×10^{-4}
η (Pa·s)	0.001	0.03	3
ΔT (K)	100	100	100
h (m)	0.01	0.01	0.01

Goals:

- Know the meaning of and how to compute the Rayleigh number.
- Understand the condition for convection initiation: $\mathrm{Ra} > \mathrm{Ra}_c$
  - For a laterally infinite fluid layer that is heated at the bottom and cooled at the top, Ra_c is $\sim$600 to 10^3.
- Be able to compute Ra for the Earth's mantle, memorize its order of magnitude and use it as an argument for the mantle convection.