Fluid Mechanics: Governing Equations

- The local and spatial form of the momentum balance equation is

$$\nabla \cdot \boldsymbol{\sigma} + \rho \, \mathbf{b} = \rho \, \frac{D\mathbf{v}}{Dt}. \tag{1}$$

  For review, see the lecture notes on continuum kinematics or watch and read the module "Eulerian Lagrangian Description" on `http://web.mit.edu/hml/ncfmf.html`.
- This equation should always hold as long as Newton's 2nd law of motion is valid no matter what material we are dealing with.
- What distinguishes solid from fluid is the "behavior" or response to loading, which is described macroscopically by a constitutive relation.
- The boundary between solid-like and fluid-like behaviors is often blurry but one way of defining fluid is to see whether a material deforms indefinitely for given shear stress.

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- If a material subjected to a fixed value of shear stress continues to deform without stopping, i.e. “flows”, the material is treated as a fluid.
- Since the amount of deformation is indefinite, a proper way of measuring deformation of fluid is to measure the **rate of deformation.**
- The rate of deformation is quantified by the **strain rate** tensor:

$$\dot{\varepsilon} = \frac{1}{2}\left[\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right] = \frac{1}{2}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T\right], \tag{2}$$

where **v** is the spatial velocity and x is the spatial coordinates.

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- In the matrix form, the strain rate tensor looks like

$$\begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\right) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1}+\frac{\partial v_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2}+\frac{\partial v_2}{\partial x_3}\right) & \frac{\partial v_3}{\partial x_3} \end{pmatrix}$$

- In the indicial notation,

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right) \quad (3)$$

- Next, we are going to make two assumptions: the flow is **incompressible** and **Newtonian.**
- An **incompressible** flow has zero volumetric strain rate:

$$\dot{e} = \text{tr}(\dot{\varepsilon}) = (\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) = \left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right) = 0. \quad (4)$$

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- The Cauchy stress tensor can always be decomposed into pressure and deviatoric stress terms.

$$\boldsymbol{\sigma} = -\mathrm{p}\,\mathbf{I} + \boldsymbol{\tau}, \tag{5}$$

where $p = -\mathrm{tr}(\boldsymbol{\sigma})/\mathrm{dim}$ and the minus sign implies that compressional pressure is positive.

- Likewise, the strain rate tensor can be decomposed into volumetric and deviatoric components.

$$\dot{\boldsymbol{\varepsilon}} = \dot{\mathrm{e}} + \dot{\boldsymbol{\epsilon}}, \tag{6}$$

- A **Newtonian** flow has linear relationship between deviatoric strain rate and stress:

$$\boldsymbol{\tau} = 2\mu\dot{\boldsymbol{\epsilon}} \tag{7}$$

where μ is called **(Newtonian) viscosity**.

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- The full constitutive relation for the incompressible Newtonian fluid becomes

$$\begin{aligned}\boldsymbol{\sigma} &= -\mathrm{p}\,\mathbf{I} + \boldsymbol{\tau} \\ &= -\mathrm{p}\,\mathbf{I} + 2\mu\dot{\boldsymbol{\epsilon}}\end{aligned} \tag{8}$$

- In the indicial notation,

$$\sigma_{ij} = -\mathrm{p}\,\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{9}$$

- Note that zero volumetric strain rate does not mean zero pressure in incompressible fluids.
- Let's rewrite the momentum balance equation (1) in terms of pressure and deviatoric stress to get the **Navier-Stokes equation**:

$$\begin{aligned}\rho\,\frac{D\mathbf{v}}{Dt} &= \nabla\cdot\boldsymbol{\sigma} + \rho\,\mathbf{b} = \nabla\cdot(-p\,\mathbf{I} + \boldsymbol{\tau}) + \rho\,\mathbf{b} \\ &= -\nabla p + \nabla\cdot\boldsymbol{\tau} + \rho\,\mathbf{b}.\end{aligned} \tag{10}$$

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- For the Newtonian fluid,

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \nabla \cdot (2\mu\dot{\epsilon}) + \rho\,\mathbf{b}. \tag{11}$$

- The notorious difficulty with solving the Navier-Stokes equation partly originates from the non-linear advection term involved in the material derivative:

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}. \tag{12}$$

- However, for steady-state flows ($\partial \mathbf{v}/\partial t = 0$) with very low speed and velocity gradient, the whole left hand side of (11) can be ignored. Creeping flows of rocks satisfy this condition.

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- So, the Navier-Stokes equation for the steady ($\partial \mathbf{v}/\partial t = 0$), smooth (i.e., $|\nabla \mathbf{v}| \ll 1$) and slow ($|\mathbf{v}| \ll 1$) Newtonian fluid becomes
$$0 = -\nabla p + \nabla \cdot (2\mu \dot{\boldsymbol{\epsilon}}) + \rho\, \mathbf{b}. \tag{13}$$
- In the indicial notation,
$$0 = -p_{,i} + \left[\mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right]_{,j} + \rho\, b_i. \tag{14}$$
where $()_{,i}$ denotes partial derivative with respect to x_i with i=1,2,3, and the indicial notation of strain rate (3) is also used.
- If viscosity (μ) is constant,
$$0 = -p_{,i} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)_{,j} + \rho\, b_i. \tag{15}$$

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- The pressure p can be decomposed into hydrostatic pressure (e.g., $\rho g x_2$) and dynamic pressure (P):

$$p = \rho g x_2 + P. \tag{16}$$

- If we further assume that the body force is solely due to gravity, then the gradient of the hydrostatic pressure and the body force terms cancel out each other.

$$-\frac{\partial \rho g x_2}{\partial x_2} + \rho g = 0. \tag{17}$$

- Then, we have only dynamic pressure term left:

$$0 = -P_{,i} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)_{,j}. \tag{18}$$

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- In a 2D case, denoting x_1 and x_2 as x and y, v_1 and v_2 as u and v, respectively, we can write the above equation as follows:
  for $i = 1$:

$$
\begin{aligned}
0 &= -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) \\
&= -\frac{\partial P}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right).
\end{aligned}
\tag{19}
$$

  for $i = 2$:

$$
\begin{aligned}
0 &= -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \right) \right) \\
&= -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} \right) + 2\mu \frac{\partial^2 v}{\partial y^2}.
\end{aligned}
\tag{20}
$$

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- The incompressibility condition for 2D becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{21}$$

- By differentiating this with respect to x, we get

$$\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial x^2}. \tag{22}$$

  Likewise, with respect to y,

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial y^2}. \tag{23}$$

- Simplifying the 2D equation with these identities, we get

$$\begin{aligned} 0 &= -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\ 0 &= -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right). \end{aligned} \tag{24}$$

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- Equations in (24) are the governing equations for the motion of a steady, smooth, slow, Newtonian fluid in 2D and the same with (6-67) and (6-68) in T&S.
- In 1D,

$$0 = -\frac{dP}{dx} + \mu \frac{d^2 u}{dx^2}. \tag{25}$$

  This is the same with (6-10) in T&S.
- The first half of the fluid mechanics chapter is just a collection of applications of these equations. Note that 1D equation is an ODE, which can be solved very easily.