Pipe Flow or Poiseuille (/pwazé:i/) Flow I

- We derived the momentum balance equation for flows that are
  - in the steady state ($\partial \mathbf{v}/\partial t = 0$),
  - little varying in the flow direction ($|\mathbf{v} \cdot \nabla \mathbf{v}| \approx 0$),
  - incompressible ($\nabla \cdot \mathbf{v} = 0$),
  - Newtonian ($\boldsymbol{\tau} = 2\mu\, \dot{\boldsymbol{\epsilon}}$)

  when gravity is the only body force. In 2D:

$$
\begin{aligned}
0 &= -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
0 &= -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right).
\end{aligned} \tag{1}
$$

In 1D,

$$
0 = -\frac{dP}{dx} + \mu \frac{d^2 u}{dx^2}. \tag{2}
$$

Can you guess the form of the equations in 3D?

Pipe Flow or Poiseuille (/pwazé:i/) Flow II

- Let's consider some simplest possible cases: 1-D channel flows.
  - Couette flow.
  - Pressure head-driven flow.
- For a flow in a perfectly circular and straight pipe, the momentum balance equation also becomes one-dimensional.
- The given geometry suggests the cylindrical coordinate system.
- v_r and v_ϕ are uniformly zero. Also, most of the spatial derivatives are zero but $\partial v_x/\partial r$ is not, where x axis coincides with the central axis of the pipe.

Pipe Flow or Poiseuille (/pwazé:i/) Flow III

- Considering the above conditions, we end up with only the x component in the cylindrical momentum balance equation:

$$0 = -\frac{dP}{dx} + \mu\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_x}{dr}\right)\right). \tag{3}$$

- Integrating (3) over r once, we get

$$r\frac{dv_x}{dr} = \frac{1}{2\mu}\left(\frac{dP}{dx}\right)r^2 + A,$$

where A is an integration constant.

- Integrating one more time after dividing both sides by r,

$$v_x = \frac{1}{4\mu}\left(\frac{dP}{dx}\right)r^2 - \frac{A}{r^2} + B.$$

Pipe Flow or Poiseuille (/pwazé:i/) Flow IV

- Since v_x should be finite at $r = 0$, A must be zero. Also we can compute B from $v_x = 0$ at $r = R$.

$$v_x = \frac{1}{4\mu}\frac{dP}{dx}\left(r^2 - R^2\right). \tag{4}$$

- The maximum velocity (u_{max}) and the volumetric flow rate (Q), which is the total volume of fluid passing a cross section per unit time, are:

$$u_{max} = -\frac{R^2}{4\mu}\frac{dP}{dx}, \tag{5}$$

$$Q = \int_0^R u\,dA = \int_0^R u\,2\pi r\,dr = -\frac{\pi R^4}{8\mu}\frac{dP}{dx}. \tag{6}$$

- By dividing Q with the area (πR^2), we get the mean velocity,

$$\bar{u} = -\frac{R^2}{8\mu}\frac{dP}{dx} = \frac{1}{2}u_{max}. \tag{7}$$

Laminar vs. Turbulent Flow I

- Watch 6 m 30 s - 11 m 40 s of this movie[1] `http://www.youtube.com/watch?v=1_oyqLOqwnI&start=390&feature=share&list=PL0EC6527BE871ABA3`.
- Behaviors of a high viscosity and low velocity flow are fundamentally different (not just in speed!) from those of a low viscosity and high velocity flow: Collectively, the former are said to be **laminar** while the latter **turbulent**.
- The term laminar means "layered" because in laminar flows, a "layer" of fluid corresponding to a certain velocity in its parabolic profile never crosses another layer of different velocity.
- Turbulent flows are, in contrast, characterized by vigorous "mixing" within the entire fluid layer. A turbulent flow becomes unsteady with random eddies.

Laminar vs. Turbulent Flow II

- **Reynolds number** is defined as

$$Re \equiv \frac{\rho v L}{\mu} = \frac{vL}{\nu}, \tag{8}$$

  where v and L are the velocity and length scale and *nu* is the *kinematic viscosity* (not the Poisson's ratio!).

- In our Poiseuille flow setting,

$$Re = \frac{\rho \bar{u} D}{\mu}, \tag{9}$$

  where $D = 2R$.

- Transition from a laminar to a turbulent flow is only empirically known to occur at $Re \approx 2200$ in case of a pipe flow.

Laminar vs. Turbulent Flow III

- It is convenient to define the **frictional factor** (equivalent to the non-dimensional pressure gradient) for explaining another difference between laminar and turbulent flows.
- The frictional factor (f) is involved in the Darcy-Weisbach equation that describes the pressure drop in a pipe flow:

$$\Delta P = -f \frac{L}{D} \frac{\rho \bar{u}^2}{2}. \tag{10}$$

  where D is the diameter of the pipe, equal to $2R$.
- In case the pressure gradient is constant so that uniformly equal to $\Delta P / L$, we can rearrange this equation to

$$f = -\frac{4R}{\rho \bar{u}^2} \frac{dP}{dx}. \tag{11}$$

Laminar vs. Turbulent Flow IV

- Substituting the mean velocity for the Poiseuille flow for $\bar{u}$, we get
$$f = \frac{64}{Re}. \tag{12}$$
- When plotted against Re, f jumps at around 2000 (see Fig. 6-7 in T&S) and follows the following empirical trend
$$f = 0.3164\, Re^{-1/4}. \tag{13}$$
- Considering the assumptions we made, the Poiseuille flow solution obviously describes a laminar flow: v_r and v_ϕ are zero therefore no "mixing" or momentum transfer is possible.
- Thus, the relation (12), based on the Poiseuille flow solution, is also valid only for a laminar pipe flow.

Laminar vs. Turbulent Flow V

- Eq. (13) holds for turbulent flows ($Re > 2000$) and indicates that higher pressure gradients are required than expected by a laminar flow theory.
- For example, to maintain the same velocity, we'll have to apply a greater pressure gradient to a low viscosity, turbulent flow than to a high viscosity, laminar flow.

[1] Again from this collection of fluid mechanics movies and notes: http://web.mit.edu/hml/ncfmf.html

Geophysical Applications of the Poiseuille Flow

1. Artesian Aquifer Flows
   - See Fig. 6-9 of T&S. The pressure gradient responsible for the flow out of a well is

$$\frac{dP}{ds} = -\frac{\rho g b}{\pi R'}, \tag{14}$$

     where s and R' are the arc length along and the radius of the circular aquifer, b is the height difference between the two ends of the aquifer.
   - According to the equation for the volumetric (laminar) flow rate (6),

$$Q = \frac{\rho g b R^4}{8 \mu R'}, \tag{15}$$

     where we set dP/ds to be equal to dP/dx and R is the radius of the aquifer.

Geophysical Applications of the Poiseuille Flow

1. Artesian Aquifer Flows

- If the flow is turbulent[2],

$$-\frac{4R}{\rho\bar{u}^2}\frac{dP}{dx} = 0.3164\left(\frac{\mu}{\rho\bar{u}\,2R}\right)^{1/4}. \qquad (16)$$

- Using the pressure gradient given in (14), we can solve for $\bar{u}$:

$$\bar{u} = \left(\frac{4\times 2^{1/4}}{0.3164}\right)^{4/7}\left(-\frac{1}{\rho}\frac{dP}{dx}\right)^{4/7} R^{5/7}\left(\frac{\rho}{\mu}\right)^{1/7}. \qquad (17)$$

- By multiplying πR^2 to the above equation and using (14), we can get the volumetric flow rate as

$$Q \approx 7.686\left(\frac{gb}{R'}\right)^{4/7} R^{19/7}\left(\frac{\rho}{\mu}\right)^{1/7}. \qquad (18)$$

[2] Note that the definition of the friction factor doesn't assume a laminar flow.

Geophysical Applications of the Poiseuille Flow

2. Magma Flow through Volcanic Pipes

- If magma is in hydrostatic state, i.e., $p_l = \rho_l g x$ where p_l is the pressure in a magma-filled pipe, ρ_l is the magma density and x is the vertical coordinate, increasing upward, it wouldn't flow.
- So, extra driving pressure is needed to make magma flow through the pipe. Here, we assume that it is just as much as needed p_l to be equal to the lithostatic pressure, p_s:

$$p_l = -\rho_l g x + \Delta P = p_s \text{ and } p_s = -\rho_s g x. \tag{19}$$

  where ρ_s is the density of lithosphere.
- So, the pressure driving magma upward would be

$$\Delta P = -(\rho_s - \rho_l) g x. \tag{20}$$

Geophysical Applications of the Poiseuille Flow

2. Magma Flow through Volcanic Pipes
   - The vertical pressure gradient is then

$$\frac{dP}{dx} = -(\rho_s - \rho_l)g \tag{21}$$

   - Once the pressure gradient is known, one can compute $\bar{u}$ and Q for laminar and turbulent cases as in the artesian aquifer problem.

2D Flows - The Stream Function

- Let's define a potential function, $\psi(x, y)$, of which spatial derivatives are the velocity components:

$$u = -\frac{\partial \psi}{\partial y} \tag{22}$$

$$v = \frac{\partial \psi}{\partial x} \tag{23}$$

- The continuity equation $\partial u/\partial x + \partial v/\partial y = 0$ is automatically satisfied.
- The momentum equations become

$$0 = \frac{\partial P}{\partial x} + \mu \left(\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3} \right) \tag{24}$$

$$0 = -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial y^2 \partial x} \right) \tag{25}$$

2D Flows - The Stream Function

- We can eliminate the pressure terms by taking partial derivative of the above equations with respect to y and x, respectively and summing them up. Then we get the following **biharmonic equation**:

$$0 = \frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = \nabla^4 \psi, \tag{26}$$

where ∇^2 is the Laplacian operator, $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

Applications

Stream functions:

- Sec. 6.10 Postglacial rebound
- Sec. 6.11 Angle of subduction
- Sec. 6.12 Diapirism
- Sec. 6.13 Folding

Others:

- Sec. 6.14 Stokes Flow
- Sec. 6.15 Plume heads and tails (Stokes flow application)
- Sec. 6.16 Pipe Flow with Heat Addition (derivation from the heat advection-diffusion eq)