Stress

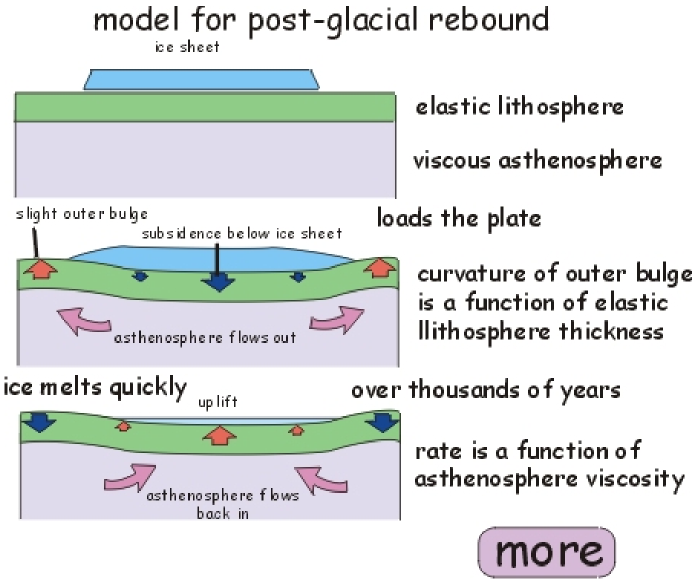
- We learned how to quantitatively describe the motion of a continuum body including its “internal deformation”, which is represented by ***strain***.
- In this lecture, we study what is the force associated with the internal deformation and how to incorporate it into the equation of force balance.
- A motion of a body is caused by two kinds of forces: Body and Surface (or contact) force.
  - Gravity governing the free fall of a billiard ball: pure body force.
  - Momentum transfer by collision with another billiard ball: (mostly) surface force.
  - Easy to find examples of deformation of continua by surface forces.

Stress

http:

//www.amazon.com/Accoutrements-12021-Stress-Cupcake/dp/B00424LEV4/?tag=cupcakefun-20

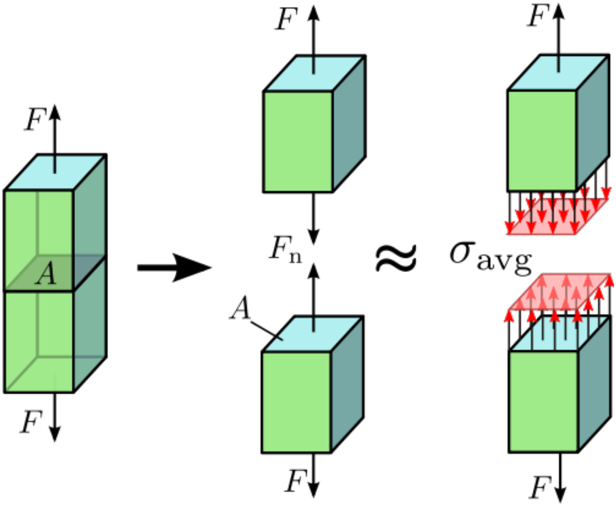
Stress



http://www.see.leeds.ac.uk/structure/dynamicearth/plates/rebound.htm

Stress

- Let's consider a continuous body that is being strained by both body and surface forces.



http://en.wikipedia.org/wiki/Stress_%28mechanics%29

Stress

- We need a quantity that represents the internal force arising in response to deformation occurring to a continuum body.
- Such a force should be additive to the body force: i.e., the surface force is also a vector.
- It is useful to consider force “densities”:
  ex) The total graviational force is given by the integration of its density:
  $$M\mathbf{g} = \int_V \rho \mathbf{g} dV,$$
  where $\rho\mathbf{g}$ is the force density.

Stress

- Generally, a body force, $\mathbf{F}_b$ is the volume integration of its density, $\mathbf{b}$:

$$\mathbf{F}_b = \int_V \mathbf{b} \, dV.$$

- Likewise, the surface force ($\mathbf{F}_s$) can also be acquired by integrating its surface density:

$$\mathbf{F}_s = \int_A \mathbf{t} \, dA.$$

  We call $\mathbf{t}$, the surface force per area, ***traction***.

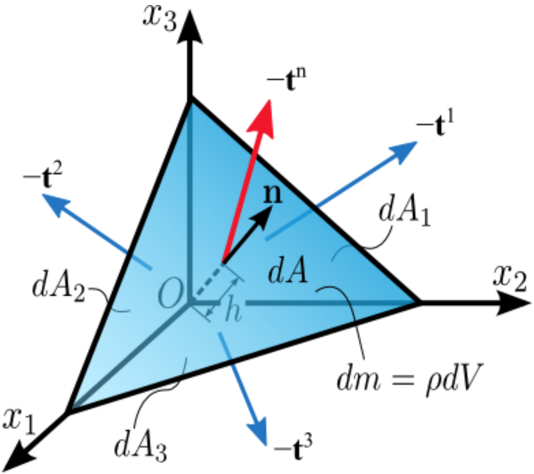
- With these force densities, we can talk about the local forces acting on a point in the body rather than on the whole body.

Stress

- Note that different tractions arise on differently oriented areas even if the “state” of the material is unchanged.
- In particular, the traction is a *linear* function of the normal vector: i.e., we want to have *net traction* by summing up tractions acting on different parts of a surface.
- This property requires the existence of a linear mapping from a normal vector to a traction vector.
- Since a rank 2 tensor can represent such a linear mapping, this relationship hints the idea of stress tensor.
- Let’s look at the reasoning leading to the concept of stress tensor more carefully.

Stress

- Cauchy's tetrahedron:



Stress

- When no body force is acting, the force equilibrium states

$$\mathbf{t}^n\, dA - \mathbf{t}^1\, dA_1 - \mathbf{t}^2\, dA_2 - \mathbf{t}^3\, dA_3 = \rho \left(\frac{h}{3}\, dA \right) \mathbf{a} \qquad (1)$$

- Since dA_i, i=1, . . . ,3 is projection of dA,

$$\begin{aligned} dA_1 &= \mathbf{n} dA \cdot \mathbf{e}_1 \\ dA_2 &= \mathbf{n} dA \cdot \mathbf{e}_2 \\ dA_3 &= \mathbf{n} dA \cdot \mathbf{e}_3 \end{aligned} \qquad (2)$$

- Substituting (2) into (1), we get

$$\mathbf{t}^n - \mathbf{t}^1(\mathbf{n} \cdot \mathbf{e}_1) - \mathbf{t}^2(\mathbf{n} \cdot \mathbf{e}_2) - \mathbf{t}^3(\mathbf{n} \cdot \mathbf{e}_3) = \rho \left(\frac{h}{3} \right) \mathbf{a} \qquad (3)$$

Note that dA has been cancelled out.

Stress

- In the limit $h \to 0$, the right hand side is identically zero. Therefore,
$$\mathbf{t}^n = \mathbf{t}^1 \; n_1 + \mathbf{t}^2 \; n_2 + \mathbf{t}^3 \; n_3. \tag{4}$$
Or,
$$\begin{aligned} t^n_1 &= t^1_1 n_1 + t^2_1 n_2 + t^3_1 n_3, \\ t^n_2 &= t^1_2 n_1 + t^2_2 n_2 + t^3_2 n_3, \\ t^n_2 &= t^1_3 n_1 + t^2_3 n_2 + t^3_3 n_3. \end{aligned} \tag{5}$$
- Eq. (4) further implies that there is a rank 2 tensor, $\boldsymbol{\sigma}$, such that
$$\mathbf{t}^n = \boldsymbol{\sigma} \; \mathbf{n}, \tag{6}$$
where the column vectors of $\boldsymbol{\sigma}$ are $\mathbf{t}^i$ $(i = 1 \ldots 3)$.

Properties of Cauchy Stress Tensor

- We call the rank 2 tensor $\boldsymbol{\sigma}$ the ***Cauchy stress tensor***.
- Note that all the considerations so far have been made with respect to the *current* (or deformed) configuration.
- When projected along the standard orthonormal basis, $\{\mathbf{e}_a\}$, we get

$$\mathbf{e}_a \cdot \boldsymbol{\sigma}\mathbf{e}_b = \sigma_{ab}.$$

- Also,

$$\begin{aligned}
\mathbf{t}_{e_1} &= \boldsymbol{\sigma}\mathbf{e}_1 = \sigma_{11}\mathbf{e}_1 + \sigma_{21}\mathbf{e}_2 + \sigma_{31}\mathbf{e}_3, \\
\mathbf{t}_{e_2} &= \boldsymbol{\sigma}\mathbf{e}_2 = \sigma_{12}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{32}\mathbf{e}_3, \\
\mathbf{t}_{e_3} &= \boldsymbol{\sigma}\mathbf{e}_3 = \sigma_{13}\mathbf{e}_1 + \sigma_{23}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3.
\end{aligned} \tag{7}$$

- From the relations (7), we can visualize traction components and stress components as in Fig. 3.3 of the handout.

Properties of Cauchy Stress Tensor

- Cauchy stress, σ, is ***symmetric***. For a proof, wait until we get to the balance principles.
- Understand the following concepts and the characteristics of the associated Cauchy stress matrix:
  - Normal and shear stress
  - Uni-, bi- and triaxial stress state
  - Pure shear stress state
  - Hydro(or litho)static stress state
  - Plane stress state

Properties of Cauchy Stress Tensor

- We make frequent use of ***invariants***, ***principal stresses*** and associated directions.
  - Principal stresses and strains and the associated principal directions are mathematically eigenvalues and eigenvectors.
  - Review the related maths here: `https://www.khanacademy.org/math/linear-algebra/alternate-bases#eigen-everything`

Coordinate Transformation by Rotation

- Vector coordinate transformation: $\mathbf{v}' = \mathbf{Q}\mathbf{v}$, where $\mathbf{Q}$ is the forward rotation matrix. For instance, a counterclockwise rotation of a point (represented by the coordinate vector) around the x_3 axis by an angle θ(radian) is represented by the following matrix:

$$\mathbf{Q} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Note that this rotation is equivalent to the rotation of the reference frame by $-\theta$ around the x_3 axis; and thus the rotation of the reference frame by θ is represented by the matrix $\mathbf{Q}^T = \mathbf{Q}(-\theta)$.

- Now, let's say a vector $\mathbf{t}$ is mapped by a rank 2 tensor, $\boldsymbol{\sigma}$ from another vector $\mathbf{n}$ in one coordinate system.

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} \tag{8}$$

Coordinate Transformation by Rotation

- When the **coordinate system is rotated** by θ, how does the rank 2 tensor transform?
- The transformed vectors and tensors are denoted as primed symbols: $\mathbf{t}'$, $\mathbf{n}'$ and $\boldsymbol{\sigma}'$. Then in the rotated coordinate system,
$$\mathbf{t}' = \boldsymbol{\sigma}'\mathbf{n}' \tag{9}$$
- Here, we have $\mathbf{t}' = \mathbf{Q}^T\mathbf{t}$ and $\mathbf{n}' = \mathbf{Q}^T\mathbf{n}$. By plugging these into Eq. (9), we get
$$\mathbf{Q}^T\mathbf{t} = \boldsymbol{\sigma}'\mathbf{Q}^T\mathbf{n}$$
Therefore
$$\mathbf{t} = (\mathbf{Q}\boldsymbol{\sigma}'\mathbf{Q}^T)\mathbf{n} \tag{10}$$

Coordinate Transformation by Rotation

- From (8) and (10),

$$\boldsymbol{\sigma} = \mathbf{Q}\boldsymbol{\sigma}'\mathbf{Q}^T$$

  Equivalently,

$$\boldsymbol{\sigma}' = \mathbf{Q}^T\boldsymbol{\sigma}\mathbf{Q} \tag{11}$$

- In a 2D case (e.g., plane stress), we get the *Mohr Transformation*:

$$\sigma'_{11} = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + \sigma_{12}\sin 2\theta \tag{12}$$

$$\sigma'_{22} = \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - \sigma_{12}\sin 2\theta \tag{13}$$

$$\sigma'_{12} = (\sigma_{22} - \sigma_{11})\sin\theta\cos\theta + \sigma_{12}\cos 2\theta \tag{14}$$

- Using this transformation, you can find the angle of a plane on which normal or shear stress is maximized.

Mohr's circle

Let's rearrange the Mohr transformation equations:

$$\sigma'_{11} = \sigma_{11}\frac{1+\cos 2\theta}{2} + \sigma_{22}\frac{1-\cos 2\theta}{2} + \sigma_{12}\sin 2\theta$$
$$\sigma'_{22} = \sigma_{11}\frac{1-\cos 2\theta}{2} + \sigma_{22}\frac{1+\cos 2\theta}{2} - \sigma_{12}\sin 2\theta$$
$$\sigma'_{12} = (\sigma_{22}-\sigma_{11})\frac{\sin 2\theta}{2} + \sigma_{12}\cos 2\theta$$

$$\sigma'_{11} = \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta$$
$$\sigma'_{22} = \frac{\sigma_{11}+\sigma_{22}}{2} - \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta$$
$$\sigma'_{12} = -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta$$

Coordinate Transformation by Rotation

$$\sigma'_{11} = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta + \sigma_{12} \sin 2\theta \qquad (15)$$

$$\sigma'_{22} = \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta + \sigma_{12} \sin 2\theta \qquad (16)$$

$$\sigma'_{12} = -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta + \sigma_{12} \cos 2\theta \qquad (17)$$

- Find an angle θ at which the shear stress (σ'_{12}) vanishes. The angle is called the *principal angle* and the corresponding normal stresses (σ'_{11} and σ'_{22}) is called the *principal stress*.
- Since $\tan(2\theta) = \tan(2\theta + \pi) = \tan(2(\theta + \pi/2))$, $\theta + \pi/2$ is also a principal angle.
- Find the principal stresses.

Mohr's circle

- Let's assume that we started with the principal axes such that $\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_2$ and $\sigma_{12} = 0$.
- After a rotation of the coordinate axes by an arbitrary value of θ, we get the following transformed stress components:

$$\sigma'_{11} = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta = -p + R \cos 2\theta,$$
$$\sigma'_{22} = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta = -p - R \cos 2\theta,$$
$$\sigma'_{12} = -\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = -R \sin 2\theta.$$

- One can also realize that

$$(\sigma'_{11} + p)^2 + \sigma'^2_{12} = R^2,$$
and
$$(\sigma'_{22} + p)^2 + \sigma'^2_{12} = R^2.$$

Mohr's circle

- In other words, since θ is arbitrary, any transformed stress components must be on a circle centered at $-p$ with a radius R.
- This circle is called the **Mohr's circle**.
- Be careful with the sign convention. Here, we are assuming the **tension positive** convention and the definition, $p = -\sigma_{kk}/n_{\text{dim}}$.
- Let's try to understand Figs. 3.14-18 of the handout.