Continuum

- Familiar with the classical Newtonian mechanics of *a system of particles* or *a rigid body*?
- A good starting point would be to see how continuum is different from those concepts.
  - Solids, liquids, gases. Is sand continuum?
  - A system of particles: particles separated by empty spaces.
  - Rigid bodies: Infinitely strong force prevents change in distance between particles.

Continuum

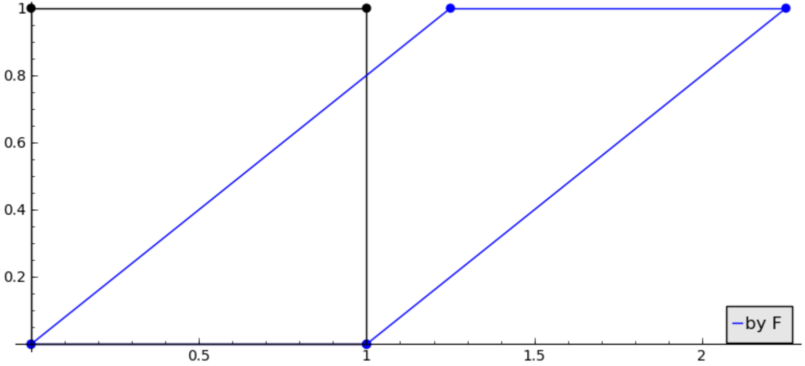
- **Continuum**: Disregard the molecular or atomic structure of matter and picture it as being **without gaps or empty spaces**.
- Another central assumption: All the mathematical functions used to describe the material are **smoothly continuous** in the entire domain or in each of finite sub-domains, so that their derivatives exist.

Description of Motion

- When we are interested in how much a solid has deformed, it makes a lot of sense to play with some "relative" measurements. For instance, when we stretch an elastic bar, it makes a lot of sense to measure the amount of extension divided by the original length.
- ex) A 2 m bar extended to 2.002 m.
  0.002 m/2 m = 0.001 or 0.1 %. This dimensionless quantity is called ***strain***.
- If we take 0.002 m, it can be either a large deformation for a short (0.002 m) bar and an infinitesimal deformation for a long (2 km) bar. But the meaning of 0.1 % strain is clearly understood.

Description of Motion

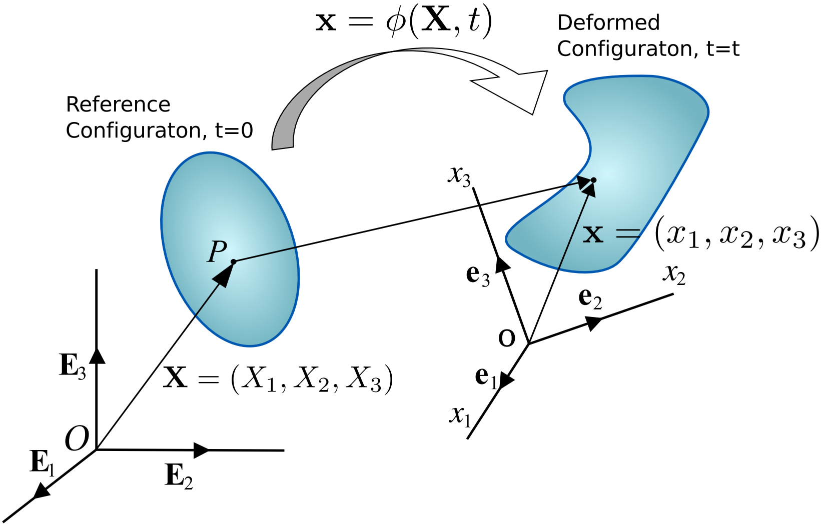
- What has changed in this case? What needs to be measured for unambiguous representation of this deformation?



Description of Motion

- Measuring deformation of a coninuum is generally not easy, particularly for non-linear and/or history-dependent materials.
- So we will stick to simple enough materials like ***linear*** elastic ones and their ***small deformation (strain)*** during most of this course.
- Now, how do we do this quantitatively?

Description of Motion



(from the continuum mechanics entry of Wikipedia)

Description of Motion

- We wish to describe the generic motion of a material body ($\mathcal{B}$), including translation and rigid body rotation as well as time dependent ones.
- To trace the motion of $\mathcal{B}$, we establish an absolutely fixed (inertial) frame of reference so that points in the Euclidean space ($\mathbf{R}^3$) can be identified by their position ($\mathbf{x}$) or their coordinates (x_i, i=1,2,3).
- The subsets of $\mathbf{R}^3$ occupied by $\mathcal{B}$ are called the ***configurations*** of the body. The *initially* known configuration is particularly called *reference configuration*.

Description of Motion

- It is conceptually important to distinguish the particles (P) of the body from their places in $\mathbf{R}^3$. The particles are physical entities - pieces of matter - whereas the places are merely positions in $\mathbf{R}^3$ in which particles may or may not be at any specific time.
- To identify particles, we label them in much the same way one labels discrete particles in classical dynamics. However, since $\mathcal{B}$ is an uncountable continuum of particles, we cannot use the integers to label them as in particle dynamics.

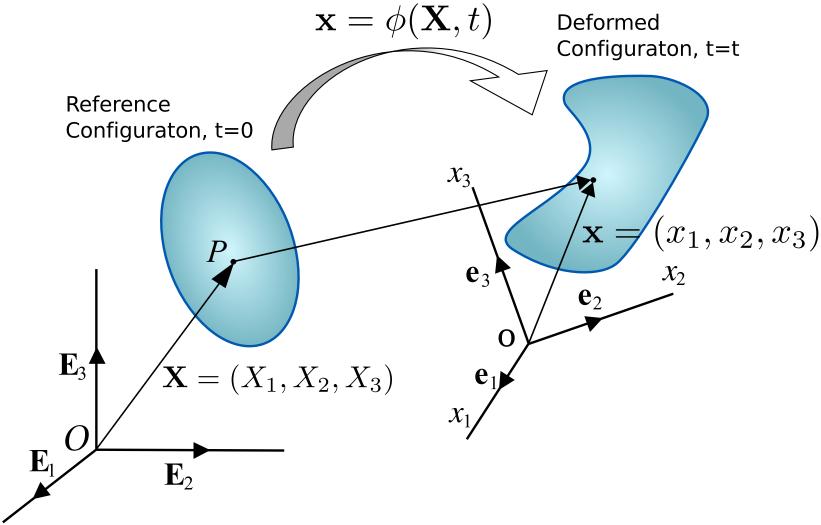
Description of Motion

- The problem is resolved by placing each particle in $\mathcal{B}$ in correspondence with an ordered triple $\mathbf{X} = (X_1, X_2, X_3)$ of *real numbers*. Mathematically, this "correspondence" is a *homeomorphism* from $\mathcal{B}$ into $\mathbf{R}^3$ and we make no distinction between $\mathcal{B}$ and the set of particle labels.
- The numbers X_i associated with particle $\mathbf{X} \in \mathcal{B}$ are called the ***material coordinates*** of $\mathbf{X}$.

Description of Motion

- For convenience, it is customary to choose the material coordinates of $\mathbf{X}$ to exactly coincide with the ***spatial coordinates***, $\mathbf{x}$, when $\mathcal{B}$ occupies its reference configuration.
- A ***motion*** of $\mathcal{B}$ is a time-dependent family of configurations, written $\mathbf{x} = \phi(\mathbf{X}, t)$. Of course, $\mathbf{X} = \phi(\mathbf{X}, 0)$.
- To prevent weird, non-realistic behaviors, we also require configurations (i.e., the mapping ϕ) to be ***sufficiently smooth*** (to be able to take derivatives), ***invertible*** (to prevent self-penetration, for instance), and ***orientation preserving*** (to prevent a mapping to a mirror image).

Description of Motion



(from the continuum mechanics entry of Wikipedia)

Description of Motion

- ***Material velocity*** of a point $\mathbf{X}$ is defined by

$$\mathbf{V}(\mathbf{X}, t) = (\partial/\partial t)\phi(\mathbf{X}, t)$$

- Velocity viewed as a function of $(\mathbf{x}, t)$, denoted $\mathbf{v}(\mathbf{x}, t)$, is called ***spatial velocity***.

$$\mathbf{V}(\mathbf{X}, t) = \mathbf{v}(\mathbf{x}, t)$$

- ***Material acceleration*** of a motion $\phi(\mathbf{X}, t)$ is defined by

$$\mathbf{A}(\mathbf{X}, t) = \frac{\partial^2 \phi}{\partial t^2}(\mathbf{X}, t) = \frac{\partial \mathbf{V}}{\partial t}(\mathbf{X}, t)$$

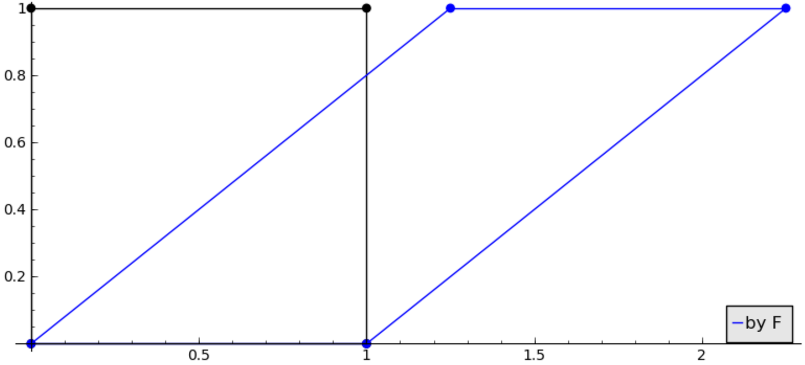
  By the chain rule,

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}$$

Description of Motion

Example:

- Find $\phi(\mathbf{X}, t)$, $\mathbf{V}$ and $\mathbf{v}$ in this simple shear motion? Assume that the top nodes are moving at a constant speed c to the right and the bottom nodes are fixed.



Description of Motion

- In general, if $Q(\mathbf{X}, t)$ is a material quantity–a given function of $(\mathbf{X}, t)$– and $q(\mathbf{x}, t) = Q(\mathbf{X}, t)$ is the same quantity expressed as a function of $(\mathbf{x}, t)$, then the chain rule gives

$$\frac{\partial Q}{\partial t} = \frac{\partial q}{\partial t} + (\mathbf{v} \cdot \nabla) q.$$

- The right-hand side is called the ***material derivative*** of q and is denoted $Dq/Dt = \dot{q}$.

- Thus Dq/Dt is the derivative of q with respect to t, holding $\mathbf{X}$ fixed, while $\partial q/\partial t$ is the derivative of q with respect to t holding $\mathbf{x}$ fixed. In particular

$$\dot{\mathbf{v}} = D\mathbf{v}/Dt = \partial \mathbf{V}/\partial t = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}.$$

Description of Motion

- ***Deformation gradient***: The 3×3 matrix of partial derivatives of ϕ with respect to $\mathbf{X}$, denoted $\mathbf{F}$ and given as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \quad \text{or} \quad F_{ij} = \frac{\partial x_i}{\partial X_j}.$$

- Some trivial cases:
  If $\mathbf{x} = \mathbf{X}$, $\mathbf{F} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix;
  if $\mathbf{x} = \mathbf{X} + ct\mathbf{E}_1$ (translation along X_1-axis with speed c), $\mathbf{F} = \mathbf{I}$. Consistent with the intuition that a simple translation is not a "deformation" of the usual sense.

- Exercises
  - What are ϕ and $\mathbf{F}$ of a "pure shear" deformation?
  - What are ϕ and $\mathbf{F}$ of a "simple shear" deformation?

- Think about why this first order partial derivative is sufficient for describing ANY general deformations.

Description of Motion

- ***Polar decomposition***: From linear algebra, we know $\mathbf{F}$ can be uniquely decomposed as

$$\mathbf{F} = \mathbf{R}\mathbf{U}(\mathbf{X}) = \boldsymbol{v}(\mathbf{x})\mathbf{R},$$

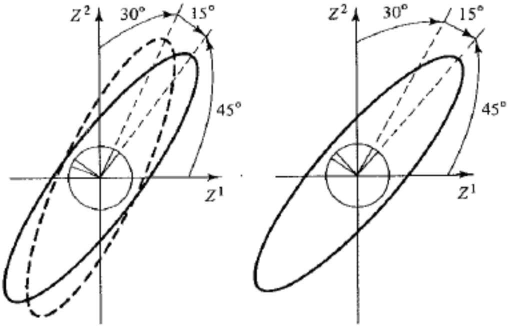
  where $\mathbf{R}$ is a proper (i.e., det($\mathbf{R}$) = +1) *orthogonal matrix* (i.e., $\mathbf{R}\mathbf{R}^T$=$\mathbf{I}$) called the *rotation*, and $\mathbf{U}$ and $\boldsymbol{v}$ are positive-definite and symmetric and called right and left *stretch tensors*.
  - Note that $\mathbf{U}$ is associated with material coordinates while $\boldsymbol{v}$ with spatial coordinates.
  - Note that symbols, $\boldsymbol{U}$ and $\boldsymbol{v}$ are also used for denoting material displacement (see below) and spatial velocity.
- $\boldsymbol{v}(\mathbf{x}) = \mathbf{R}\mathbf{U}\mathbf{R}^T$
  - $\mathbf{R}^T : \mathbf{x} \longrightarrow \mathbf{X}$
  - $\mathbf{U} : \mathbf{X} \longrightarrow \mathbf{X}$
  - $\mathbf{R} : \mathbf{X} \longrightarrow \mathbf{x}$

Description of Motion

- In general $\mathbf{U} \neq \boldsymbol{v}$.
- $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$ and $\boldsymbol{v} = \sqrt{\mathbf{F}\mathbf{F}^T}$. Furthermore, we call $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ the *right Cauchy-Green tensor* and $\mathbf{b} = \mathbf{F}\mathbf{F}^T = \boldsymbol{v}^2$ is the *left Cauchy-Green tensor*.
- Since $\mathbf{U}$ and $\boldsymbol{v}$ are *similar*, their eigenvalues are equal;
- since $\mathbf{U}$ and $\boldsymbol{v}$ are positive definite and symmetric, their eigenvalues are real and positive.
- These eigenvalues are called the ***principal stretches***.
- The deviation of principal stretches from unity measures the amount of *strain* in a deformation. Analogy can be found in the earlier simplistic example: 2.002 m/ 2 m = 1.001. Here, 0.001 is the “deviation from the unity” and represents the actual deformation.

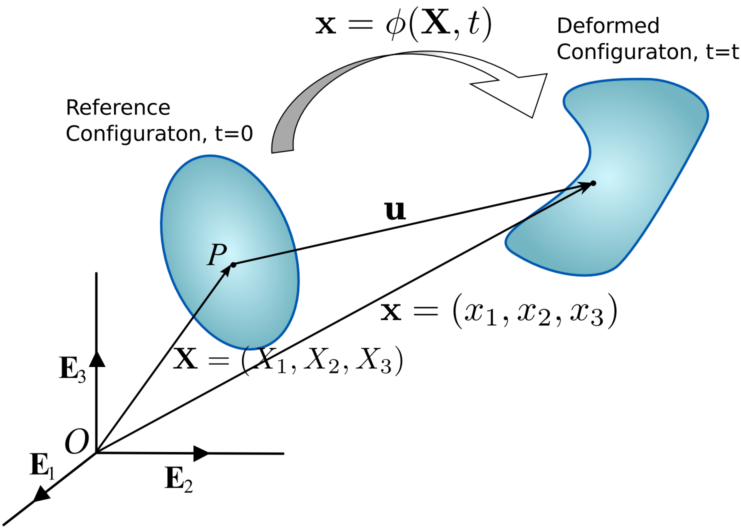
Description of Motion

- The meaning of the polar decomposition is that a deformation is locally given to first order by a rotation followed by a stretching by amounts corresponding to eigenvalues along three principal directions or vice versa.



(Fig. 1.3.1 in *Mathematical Foundations of Elasticity* (Marsden and Hughes, Dover, 1994))

Description of Motion



(modified from the continuum mechanics entry of Wikipedia)

Description of Motion

- ***Displacement*** is denoted $\mathbf{U}(\mathbf{X})$ and defined as

$$\mathbf{U}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$$

  - $\mathbf{U}(\mathbf{X}, t) = \mathbf{U}(\phi^{-1}(\mathbf{x}, t), t) = \mathbf{u}(\mathbf{x}, t)$.
  - Like the material and spatial velocity, material and spatial displacements represent the same vector field (i.e., functions returning the same numerical values for given $\mathbf{x}$ and $\mathbf{X}$ that are related by $\mathbf{x} = \phi(\mathbf{X})$.

- Since $\mathbf{x} = \mathbf{U} + \mathbf{X}$,

$$\mathbf{F} = \mathbf{I} + \frac{\partial \mathbf{U}}{\partial \mathbf{X}}.$$

- Then, $\mathbf{C}$, the right Cauchy-Green tensor, becomes

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{I} + \frac{\partial \mathbf{U}}{\partial \mathbf{X}} + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right)^T + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{X}}\right)^T \frac{\partial \mathbf{U}}{\partial \mathbf{X}}$$

Description of Motion

- Note that the rotational part (**R**) is not involved according to this definition. So, **C** is all about stretches.
- Green's (material or Lagrangian) strain tensor ("deviation from the unity"):

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$

- Note that

$$(ds)^2 = dx_i dx_i = F_{ij} dX_j F_{ik} dX_k = dX_j F_{ji}^T F_{ik} dX_k$$
$$= d\mathbf{X} \cdot (\mathbf{F}^T \cdot \mathbf{F}) \cdot d\mathbf{X} = d\mathbf{X} \cdot \mathbf{C} \cdot d\mathbf{X},$$

and

$$(dS)^2 = d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{X} \cdot \mathbf{I} \cdot d\mathbf{X}.$$

Description of Motion

- From the previous result, we realize that Green's strain tensor quantifies the change in the square of the length of the material vector $d\mathbf{X}$.

$$ds^2 - dS^2 = 2d\mathbf{X}\cdot\mathbf{E}\cdot d\mathbf{X} \quad \text{or} \quad dx_i dx_i - dX_i dX_i = 2\, dX_i\, E_{ij}\, dX_j.$$

- With further linearization, i.e., dropping the quadratic term under the assumption of infinitely small displacements, we get the familiar form of the **(small or infinitesimal) strain tensor** (ε):

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\frac{\partial\mathbf{U}}{\partial\mathbf{X}} + \left(\frac{\partial\mathbf{U}}{\partial\mathbf{X}}\right)^T\right] \text{ or } \varepsilon_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$$

- Also note that the following decomposition is always possible:

$$\frac{\partial\mathbf{U}}{\partial\mathbf{X}} = \frac{1}{2}\left[\frac{\partial\mathbf{U}}{\partial\mathbf{X}} + \left(\frac{\partial\mathbf{U}}{\partial\mathbf{X}}\right)^T\right] + \frac{1}{2}\left[\frac{\partial\mathbf{U}}{\partial\mathbf{X}} - \left(\frac{\partial\mathbf{U}}{\partial\mathbf{X}}\right)^T\right]$$

The second term represents "(rigid body) rotation".

Description of Motion

- Strain and rotation, only when combined together, describe the entire motion. Then, why do we care so much about strain and only occasionally about rotation?
  - The answer is that only strain is related to stress. More on this point later.
- ***Principal strains***, eigenvalues of a small strain tensor, have the same meaning with principal streches.
- The trace of strain (ε_{ii}) is called ***dilatation*** and often denoted e.
- ***Invariants*** of a strain tensor are all often used in various contexts. Dilatation is, for instance, the first invariant.

Description of Motion

Invariants: Three coefficients of the characteristic equation of a rank-2 tensor (**T**).

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0.$$

$$\lambda^3 - \mathrm{I}_{\mathbf{T}}\lambda^2 + \mathrm{II}_{\mathbf{T}}\lambda - \mathrm{III}_{\mathbf{T}} = 0,$$

$$\begin{aligned} \mathrm{I}_{\mathbf{T}} &= T_{11} + T_{22} + T_{33}, \\ \mathrm{II}_{\mathbf{T}} &= T_{11}T_{22} + T_{11}T_{33} + T_{22}T_{33} \\ &\quad - T_{12}T_{21} - T_{13}T_{31} - T_{23}T_{32} \\ &= \frac{1}{2}(\mathbf{T} : \mathbf{T} - \mathrm{I}_{\mathbf{T}}^2), \\ \mathrm{III}_{\mathbf{T}} &= \det \mathbf{T}. \end{aligned}$$

Exercises

- What are $\mathbf{E}$ and ε of a pure shear deformation?
- What are $\mathbf{E}$ and ε of a simple shear deformation?
- Find $\mathbf{F}$, $\mathbf{E}$ and ε for the following motion:

$$x(\mathbf{X}, t) = X(1 + at)\cos\frac{\pi t}{2} - Y(1 + bt)\sin\frac{\pi t}{2}$$
$$y(\mathbf{X}, t) = X(1 + at)\sin\frac{\pi t}{2} + Y(1 + bt)\cos\frac{\pi t}{2}$$

- Find $\mathbf{F}$, $\mathbf{E}$ and ε for the following motion:

$$x(\mathbf{X}, t) = X^2(1 + at)\cos\frac{\pi t}{2} - Y^2(1 + bt)\sin\frac{\pi t}{2}$$
$$y(\mathbf{X}, t) = X^2(1 + at)\sin\frac{\pi t}{2} + Y^2(1 + bt)\cos\frac{\pi t}{2}$$