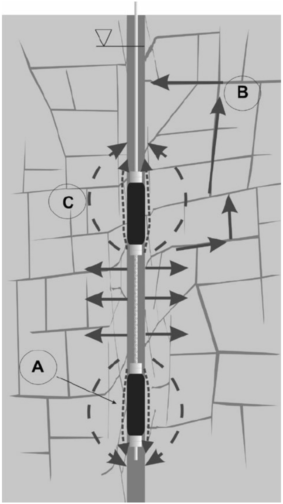
How to measure stress and strain

- Measureing stress: Overcoring
  - Can measure various components of stress (see. Fi.g 2.17 in T&S)
  - Length of the hole is limited to 1 m so it is necessary to drill the holes in mines for measurements at greater depths.

How to measure stress and strain

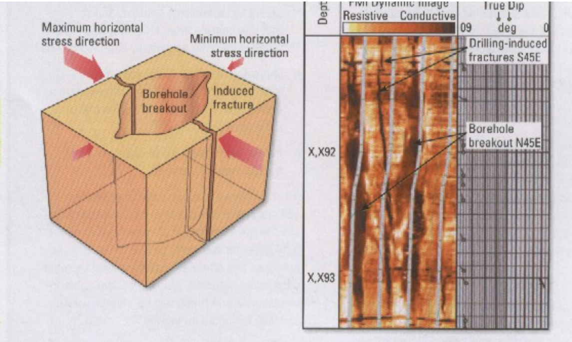
- Measureing stress: Hydrofracturing
  - See. Fig. 2.18 in T&S and understand breakdown pressure (p_b), instantaneous shut-in pressure (ISIP), and the time evolution of pressure.
  - Can measure the magnitude of σ_{min}, the minimum horizontal stress if the fractures are vertical.
  - In general, fracture planes would be perpendicular to the minimum principal stress.
  - Understand Fig. 2.19 in T&S.



Quinn et al., Hydrogeol. J., 2012.

DOI:10.1007/s10040-012-0893-8

How to measure stress and strain

- Measureing stress: Wellbore breakouts
- Provide the orientations of the maximum and minimum horizontal stresses but not always their magnitudes.
- 

https://www.spec2000.net/23-fracorientation.htm

How to measure stress and strain

- Measureing strain: Requires accurate measurements of distance between benchmarks/stations.
- Triangulation
- SLR, VLBI
- GPS
- InSAR

How to Relate Stress and Strain

- We have considered deformation of continua and some of the balance laws in them.
- These kinematics and mechanics apply to all the continuous media. Then, where do the characteristics of individual material come from?
- Properties unique to a certain material are determined by the material's internal constitution or physical make-up. The quantitative expressions for such internal constitution are called ***constitutive equations / laws / relations / models***.
- The role of constitutive relations is to relate stress (a quantity needed for stating the linear momentum balance law) and strain (a quantity used to describe motion of continua).

Linear Elasticity

- Hooke's law for a 1 dimensional mass-spring system:

$$F = -kx$$

- If no damping force acts on it, the system is *conservative*, meaning by definition that there is a potential function $U(x)$ such that $F = -\nabla U$.
- In this 1D example, integration to get U is straightforward and $U = \frac{1}{2}kx^2$.

Linear Elasticity

- A material is called ***ideally elastic*** when a body formed of the material recovers its original form completely upon removal of the forces causing the deformation, and there is a one-to-one relationship between the state of stress and the state of strain, for a given temperature.
- The one-to-one relationship precludes behaviors like creep at constant load or stress relaxation at constant strain.
- The classical elastic constitutive equations, often called the ***generalized Hooke's law***, are nine equations expressing the stress components as linear homonenous (i.e., all the terms are of the same power) functions of the nine strain components:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{1}$$

Linear Elasticity

- The rank 4 tensor, C_{ijkl}, has $81 (= 3^4)$ components.
- However, recall that stress and strain tensor are symmetric: i.e., $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{kl} = \varepsilon_{lk}$.
- Thus,
$$C_{ijkl} = C_{jikl} \text{ and } C_{ijkl} = C_{ijlk}.$$
- We further consider the case in which the material is ***elastically isotropic***, i.e., there are no preferred directions in the material. Then, the elastic constants (C_{ijkl}) must be the same at a given particle for all possible choices of rectangular Cartesian coordinates in which stress and strain components are evaluated.

Linear Elasticity

- The most general rank 4 tensor that satisfy all of the above symmetry and isotropy conditions is

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2}$$

(see Malvern Sec. 6.1 and 6.2 for further details.)

- The constitutive relation becomes

$$\sigma_{ij} = \left[\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right]\varepsilon_{kl} \tag{3}$$

- Finally, after some simplification, we reach the isotropic generalized Hooke's law:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \tag{4}$$

where λ and μ are called Lamé's constants.

Linear Elasticity

- By setting $i = j$ in Eq. (4), we find

$$\sigma_{ii} = (3\lambda + 2\mu)\varepsilon_{ii} \tag{5}$$

- By substituting $\varepsilon_{kk} = \sigma_{kk}/(3\lambda + 2\mu)$ (from Eq. (5)) into Eq. (4), we obtain

$$\varepsilon_{ij} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\sigma_{kk}\delta_{ij} + \frac{1}{2\mu}\sigma_{ij} \tag{6}$$

- Recall the definitions of Young's modulus and Poisson's relation:
  - Hooke's law: $\sigma_{xx} = E\varepsilon_{xx}$, etc, where E is the Young's modulus.
  - Poisson's relation: e.g., $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx} = -\frac{\nu}{E}\sigma_{xx}$, where ν is the Poisson's ratio.

Linear Elasticity

- From these relations, we derive the following for 3D:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \\
\varepsilon_{xy} &= \frac{1}{2\mu}\sigma_{xy}, \text{ etc.}
\end{aligned} \tag{7}$$

- The above set of equations can be generalized to the following indicial notation:

$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij} \tag{8}$$

Linear Elasticity

- It is also convenient to decompose the stress and strain tensor into deviatoric and volumetric parts:

$$
\begin{aligned}
s_{ij} &= \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \\
\epsilon_{ij} &= \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}
\end{aligned}
\tag{9}
$$

- Then, the whole relationship can be expressed by the two equations:

$$s_{ij} = 2\mu\epsilon_{ij} \text{ and } p = -Ke, \tag{10}$$

where $p = -\sigma_{kk}/3$ is the pressure, $e = \varepsilon_{kk}$ si the volume strain, and K is the bulk modulus, related to Lamé's constants by the relation

$$K = \lambda + \frac{2}{3}\mu \tag{11}$$

Linear Elasticity

- In the case of isotropic elasticity, all the elastic constants (λ, μ, E, ν, K) can be expressed in terms of just two independent constants.
- μ, the shear modulus, is often denoted as G.
- Poisson's ratio: Which is better for wine bottle, cork or rubber?
  Negative Poisson's ratio?
  `http://www.youtube.com/watch?v=HJ1Ck6FIqwU`

Linear Elasticity: Some simple cases

- Dilatation and incompressibility
  Lithostatic condition: Diviatroic stresses are zero and $\sigma_1 = \sigma_2 = \sigma_3 = \rho g h$, where h is depth, negative downward.
- Pure shear and simple shear
- Uniaxial stress: $\sigma_1 \neq 0$, $\sigma_2 = 0$ and $\sigma_3 = 0$.
- Uniaxial strain: $\varepsilon_1 \neq 0$, $\varepsilon_2 = 0$ and $\varepsilon_3 = 0$.
  - Sedimentation
  - Erosion

Linear Elasticity: Plane Stress/Strain I

- Stress state in which $\sigma_{3j} = 0$ with $j = 1 \ldots 3$.
- In terms of principal stress, $\sigma_1 \neq 0$, $\sigma_2 \neq 0$ and $\sigma_3 = 0$.
- From Eq. (8),

$$
\begin{aligned}
\varepsilon_1 &= \frac{1}{E}(\sigma_1 - \nu\sigma_2) \\
\varepsilon_2 &= \frac{1}{E}(\sigma_2 - \nu\sigma_1) \\
\varepsilon_3 &= -\frac{\nu}{E}(\sigma_1 + \sigma_2)
\end{aligned}
\tag{12}
$$

- Let's assume that the lithosphere is in a lithostatic condition.

Linear Elasticity: Plane Stress/Strain II

- Now, if deviatoric stresses in the plane stress condition are applied, of which two horizontal principal stresses are equal in magnitude,

$$\Delta\sigma_1 = \Delta\sigma_2 \neq 0 \text{ but } \Delta\sigma_3 = 0.$$

- Then, we get

$$\begin{aligned} \varepsilon_1 = \varepsilon_2 &= \frac{1-\nu}{E}\Delta\sigma_1 \\ \varepsilon_3 &= -\frac{2\nu}{E}\Delta\sigma_1. \end{aligned} \tag{13}$$

- Plane strain: $\varepsilon_1 \neq 0$, $\varepsilon_2 \neq 0$ and $\varepsilon_3 = 0$.