Balance Laws: Transport Theorem

- Reminder: material time derivative
  - Time derivative of a quantity Q in the referece configuration:

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} = \frac{\partial Q(\mathbf{X}, t)}{\partial t}.$$

  - Time derivative of the same quantity expressed in the current configuration: Since $q(\mathbf{x}(\mathbf{X}, t), t) = Q(\mathbf{X}, t)$,

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + (\mathbf{v} \cdot \nabla) q. \tag{1}$$

- Reynolds transport theorem

$$\begin{aligned} \frac{D}{Dt} \int_{v(t)} f \, dv &= \int_{v(t)} \left(\frac{Df}{Dt} + f \nabla \cdot \mathbf{v} \right) dv \\ &= \int_{v(t)} \left(\frac{\partial f}{\partial t} + \nabla \cdot (f \mathbf{v}) \right) dv \end{aligned} \tag{2}$$

Balance Laws: Transport Theorem

- Proof of the transport theorem:
  By change of variables, and differentiating under the integral sign,

$$\frac{d}{dt}\int_{v(t)} f\,dv = \frac{d}{dt}\int_V f(\phi(\mathbf{X},t),t)\,J(\mathbf{X},t)\,dV$$
$$= \int_V \left[J(\mathbf{X},t)\,\frac{d}{dt}f(\phi(\mathbf{X},t),t) + f(\phi(\mathbf{X},t),t)\frac{d}{dt}J(\mathbf{X},t)\right] dV.$$

  where J is the Jacobian determinant, $\det(\partial\mathbf{x}/\partial\mathbf{X})$.
  Changing variables back to $\mathbf{x}$, we get

$$\frac{d}{dt}\int_{v(t)} f\,dv = \int_V \left[\dot{f}(\phi(\mathbf{X},t),t) + f(\phi(\mathbf{X},t),t)\frac{\dot{J}}{J}\right] J\,dV$$
$$= \int_{v(t)} \left[\dot{f}(\mathbf{x},t) + f(\mathbf{x},t)\,\nabla\cdot\mathbf{v}\right]\,dv$$

Balance Laws: Transport Theorem

- The physical meaning is that the rate of change of a quantity contained within (or integrated over) the current configuration is equal to the volume integration over the current configuration of the time rate of change of the quantity and its net flux associated with the motion of material.

Balance Laws: Mass balance (or conservation)

- Mass conservation (material is neither created nor lost during deformation):

$$\frac{\partial}{\partial t}\int_V R\, dV = \frac{D}{Dt}\int_{v(t)} \rho(\mathbf{x}, t) dv = 0 \tag{3}$$

- By Reynolds transport theorem, we get

$$\int_{v(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right] dv = 0. \tag{4}$$

- Since Eq. (4) should hold for arbitrary subset of the body, the integrand itself must vanish everywhere if it is *continuous* almost everywhere. Then, we get the usual differential equation form of the mass conservation principle:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{5}$$

Balance Laws: Force Balance (or conservation of linear momentum)

- For a mass particle:

$$\frac{d\mathbf{p}}{dt} = \sum_i \mathbf{F}_i, \tag{6}$$

  where $\mathbf{p}$ is the linear momentum of the particle and $\mathbf{F}_i$ is the i-th force acting on it.

- For a continuous body,

$$\frac{d}{dt}\int_{v(t)} \rho\, \mathbf{v}\, dv = \int_{v(t)} \mathbf{b}\, dv + \int_{\partial v(t)} \mathbf{t}\, dS \tag{7}$$

  or if we introduce Cauchy stress into the above equation, we get

$$\frac{d}{dt}\int_{v(t)} \rho\, \mathbf{v}\, dv = \int_{v(t)} \mathbf{b}\, dv + \int_{\partial v(t)} \boldsymbol{\sigma}\mathbf{n}\, dS \tag{8}$$

Balance Laws: Force Balance (or conservation of linear momentum)

- By the Gauss's theorem[1],

$$\int_{\partial v(t)} \boldsymbol{\sigma} \mathbf{n} \, dS = \int_{v(t)} \nabla \cdot \boldsymbol{\sigma} \, dv. \tag{9}$$

- Therefore, (8) becomes

$$\int_{v(t)} (\nabla \cdot \boldsymbol{\sigma} + \mathbf{b}) \, dv - \frac{d}{dt} \int_{v(t)} \rho \, \mathbf{v} \, dv = 0. \tag{10}$$

- We apply Reynolds transport theorem to the second term on the left hand side of Eq. (10). Interestingly, we get the following identity:

$$\frac{d}{dt} \int_{v(t)} \rho \, \mathbf{v} \, dv = \int_{v(t)} \rho \, \frac{d\mathbf{v}}{dt} \, dv.$$

[1] For proof, see `https://www.khanacademy.org/video/divergence-theorem-proof-part-1`

Balance Laws: Force Balance (or conservation of linear momentum)

- Plugging the previous identity into Eq. (10), we get

$$\int_{v(t)} \left[\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} - \rho \frac{d\mathbf{v}}{dt} \right] dv = 0. \tag{11}$$

- Since Eq. (11) should hold not only for the entire volume but also for any arbitrary subset of the body, the integrand itself should be zero IF it is a continuous function. Consequently, we obtain the *local* equation of motion or force balance:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \rho \frac{d\mathbf{v}}{dt}. \tag{12}$$

Balance Laws: Force Balance (or conservation of linear momentum)

- If the motion is not time dependent, meaning either a static equilibrium (all the velocities are zero) or a steady state (all the velocities are constant, possibly non-zero), the inertial term of Eq. (13) is zero and the local equation of motion becomes
$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0. \tag{13}$$
This is the most frequently encountered form in geodynamics.

Balance Laws: Force Balance (or conservation of linear momentum)

- We can change the coordinates to those of the reference configuration and get the corresponding set of equations. Although very important in non-linear continuum mechanics, we are not going to cover them here.
- Those who are interested might want to read
  - Belytschko, T., W. K. Liu, B. Moran, and K. I. Elkhodary (2014), Nonlinear Finite Elements for Continua and Structures, 2nd ed., John Wiley & Sons, Ltd.
  - Holzapfel, G. A. (2000), Nonlinear solid mechanics : a continuum approach for engineering, Wiley, Chichester ; New York.
  - Malvern, L. E. (1969), Introduction to the Mechanics of a Continuous Medium, Prentice-Hall, Inc., Upper Saddle River, New Jersey.