CERI 8315 Geodynamics:

Homework 2

Due in class on Wednesday, October 4, 2023

1. (20 points) Let's take as a reference configuration a rectangle in $\mathbb{R}^2$, $\{\mathbf{X}|\mathbf{X} \in [0, a] \times [0, b]\}$. A motion, $\mathbf{x} = \phi(\mathbf{X}, t)$ is given as follows:

$$x_1(\mathbf{X}, t) = X_1 + \frac{ctX_1}{a}$$
$$x_2(\mathbf{X}, t) = X_2 + \frac{ctX_2}{b},$$

where $a,\ b,\ c$ are positive contants.

   (a) (3 points) Plot deformed configurations for $t = 1, 2, 3$ when $a = b = 1$ and $c = 0.1$.

   For the rest, we assume that a and b are arbitrary positive constants and that $ct \ll 1$.

   (b) (3 points) Find the spatial velocity field, $\mathbf{v}(\mathbf{x}, t)$.

   (c) (3 points) Find the (small) strain tensor, $\boldsymbol{\varepsilon}$, and the (small) strain rate tensor, $\dot{\boldsymbol{\varepsilon}}$.

   Recall the global conservation of mass stating $\int_{V_0} \rho_0(\mathbf{X}) dV = \int_{\mathrm{v}(t)} \rho(\mathbf{x}, t) d\mathrm{v}$, where ρ_0 and V_0 are the density and volume (*area* in our case) of the reference configuration, respectively, and $\rho(\mathbf{x}, t)$ and $\mathrm{v}(t)$ are those of a current configuration at time t. Also note that $\int_{\mathrm{v}(t)} \rho(\mathbf{x}, t) d\mathrm{v} = \int_{V_0} \rho(\mathbf{x}(\mathbf{X}, t), t)|J| dV$, where $|J|$ is the determinant of the Jacobian matrix (equivalent to the deformation gradient tensor):

$$|J| = \det \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right).$$

   Therefore, we get the following general identity: $\rho|J| = \rho_0$.

   (d) (3 points) For the given motion and a constant ρ_0, find the expression for the spatial density $\rho(\mathbf{x}, t)$.

   (e) (4 points) Find $\partial\rho(t)/\partial t$ and $\nabla \cdot (\rho\mathbf{v})$ to verify the spatial form of the local mass conservation equation.

   (f) (4 points) Finally, verify that $\int_{\mathrm{v}(t)} \nabla \cdot (\rho\mathbf{v}) dV = \int_{\partial\mathrm{v}(t)} \rho\mathbf{v} \cdot \mathbf{n} dS$. Here, boundaries are understood as the set of *edges* of a current configuration.

2. (5 points) Assume a plane stress state in the x_1-x_2 plane. Using the rotational transformation formulae for the components of the Cauchy stress tensor, find angle(s) at which diagonal components vanish and a condition for such an angle to exist.

3. (35 points) Solve Problem 2.19, 2.21, 2.22, 3.1, 3.2, 3.3 and 3.4 (5 points each) in the handout from Geodynamics (Turcotte and Schubert, 3rd ed., Cambridge University Press).

4. (20 points) Derive the symmetry of the Cauchy stress tensor starting from the global form of balance of angular momentum. Feel free to refer to the online resources or the reference books listed in the syllabus. However, your submitted solution should be self-contained.

5. (5 points) Show the following identity using the mass balance equation:

$$\rho\frac{D\Phi}{Dt} = \frac{\partial(\rho\Phi)}{\partial t} + \nabla\cdot(\rho\Phi\mathbf{u}),$$

   where $\Phi = \Phi(\mathbf{x}, t)$ is a spatial scalar field.

6. (a) (5 points) By expanding the Stokes equation ($\nabla\cdot\boldsymbol{\sigma} + \mathbf{b} = 0$) component wise, find the Cauchy stress in equilibrium with the body force, $\mathbf{b} = -\rho g\mathbf{e}_3$, where ρ is a density and g is the magnitude of the gravitational acceleration assumed to be constant. Use the Cartesian coordinate system with x_3 pointing upward vertical.

   (b) (5 points) Consider the Cauchy stress distribution for a continuum in equilibrium given with reference to a rectangular x_1, x_2, x_3-coordinate system. The components of the Cauchy stress tensor $\boldsymbol{\sigma}$ are given in the form

$$\boldsymbol{\sigma} = \begin{bmatrix} x_1x_2 & x_1^2 & -x_2 \\ x_1^2 & 0 & 0 \\ -x_2 & 0 & x_1^2+x_2^2 \end{bmatrix}.$$

   Find the body force $\mathbf{b}$ that acts on this continuum.

   (c) (5 points) Plot the distribution of the second invariant of the Cauchy stress in (b) over a region on the $x_1 - x_2$ plane, $[-5, 5] \times [-5, 5]$.

7. (10 points) Derive the symmetry of the Cauchy stress tensor starting from the global form of balance of angular momentum. Feel free to refer to the online resources or the reference books listed in the syllabus. However, your submitted solution should be self-contained.