CERI 8353 Geodynamics:

Homework 1

Due in class on Wednesday, September 13, 2023

1. (14 points) Assume a square in $\mathbb{R}^2$, $[-1,1]\times[-1,1]$, as a reference configuration and find $\boldsymbol{\phi}$, $\mathbf{U}$(material displacement), $\mathbf{V}$(material velocity), $\mathbf{v}$(spatial velocity), $\mathbf{F}$, $\mathbf{E}$, $\boldsymbol{\varepsilon}$for the following deformations:

   (a) (7 points) A pure shear deformation occurring at a constant rate, c, without translation of the point $(0,0)$.

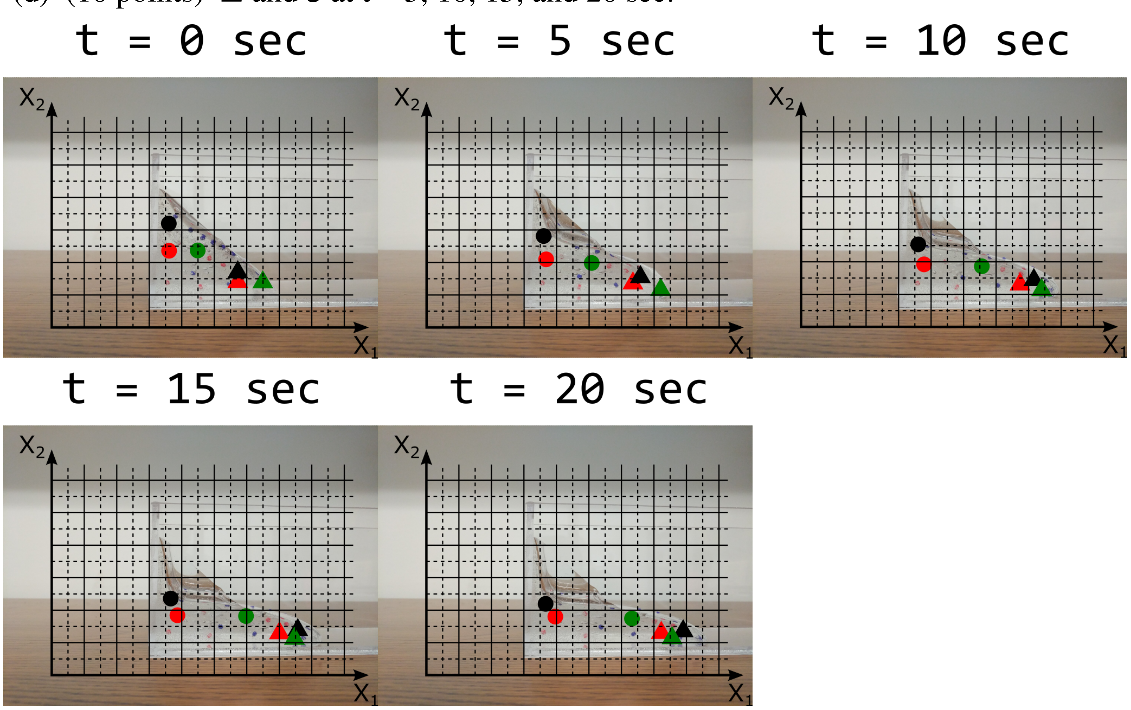
   (b) (7 points) A simple shear deformation in which the top edge moving at a velocity, $c\sin(\omega t)$, and maintaining the top and bottom edges horizontal.

2. (40 points) (a) (10 points) Tabulate the positions of the six material points (circles and triangles in red, black and green) in the reference configuration (t = 0 sec) and their corresponding spatial points in the deformed configurations at t = 5, 10, 15, and 20 sec. The solid grid lines are 1 cm apart.

   (b) (10 points) Find $\mathbf{U}$ (material displacement) at the six material points at t = 5, 10, 15, and 20 sec.

   (c) (10 points) Find $\mathbf{F}$ at the red symbols at t = 5, 10, 15, and 20 sec.

   (d) (10 points) $\mathbf{E}$ and $\boldsymbol{\varepsilon}$ at t = 5, 10, 15, and 20 sec.



3. (10 points) As a reference configuration, take a unit square constructed by the standard bases of $\mathbb{R}^2$. Pick a general (i.e., other than identity, simple shear and pure shear) $\mathbf{F}$ that is constant in space; and plot the deformed configurations mapped by
   (a) (5 points) $\mathbf{U}$ and then by $\mathbf{R}$ from the $\mathbf{RU}$ decomposition of $\mathbf{F}$.
   (b) (5 points) $\mathbf{R}$ and then by $\mathbf{v}$ from the $\mathbf{vR}$ decomposition of $\mathbf{F}$.

4. (10 points) Let $\boldsymbol{\varepsilon}$ be a small strain matrix in $\mathbf{R}^3$.
   (a) (5 points) Derive three invariants of $\boldsymbol{\varepsilon}$ in terms of its components ε_{ij}.
   (b) (5 points) $\boldsymbol{\epsilon} = \boldsymbol{\varepsilon} - \frac{1}{3}\text{tr}(\boldsymbol{\varepsilon})\mathbf{I}$ is called the *deviatoric* strain. Derive the formulae for three invariants of the deviatoric strain matrix. What can you say about the nature of the deviatoric strain?

5. (10 points) For the displacement gradient matrix

$$\frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \begin{pmatrix} 4 & -1 & 0 \\ 1 & -4 & 2 \\ 4 & 0 & 6 \end{pmatrix} \times 10^{-3},$$

   determine
   (a) (2 points) the small strain ($\boldsymbol{\varepsilon}$) and rotation ($\boldsymbol{\omega}$) matrices
   (b) (2 points) the volumetric strain and the deviatoric strain matrix
   (c) (3 points) three strain invariants
   (d) (3 points) the principal strains and their directions

6. (15 points) $\mathbf{u}$ and $\mathbf{v}$ are three dimensional vectors. Equivalent indicial notations are u_i with $i = 1, 2, 3$; and v_j with $j = 1, 2, 3$.
   - Dot product of two vectors: $\mathbf{u} \cdot \mathbf{v} = u_i v_i$.
   - Wedge product of two vectors: $\mathbf{u} \wedge \mathbf{v} = \epsilon_{ijk}\mathbf{e}_i u_j v_k$, where $\mathbf{e}_i$ is the i-th standard basis (i.e., $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$) and ϵ_{ijk} is the permutation tensor (a.k.a. the Levi-Civita tensor): $+1$ if $ijk \in \{123, 231, 312\}$, -1 if $ijk \in \{213, 132, 321\}$ and 0 for any repeating indices (e.g., 112, 212, etc).

   (a) (2 points) Recalling that repeating indices mean the summation over the entire range of index, expand the indicial notation for $\mathbf{u} \cdot \mathbf{v}$. Is the resultant quantity a scalar or a vector?
   (b) (3 points) Expand the indicial notation of the wedge product of two vectors. Is the resultant quantity a scalar or a vector?
   (c) (5 points) Show that $(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w})$, where $\mathbf{w}$ is another three-dimensional vector.
   (d) (5 points) Show that $\epsilon_{ijk}\epsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}$ using the following facts:
      - δ_{ij} is the Kronecker delta: 1 if $i = j$; 0 if $i \neq j$.

- $$\epsilon_{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix},$$

  where $|\cdot|$ is the determinant of a matrix.