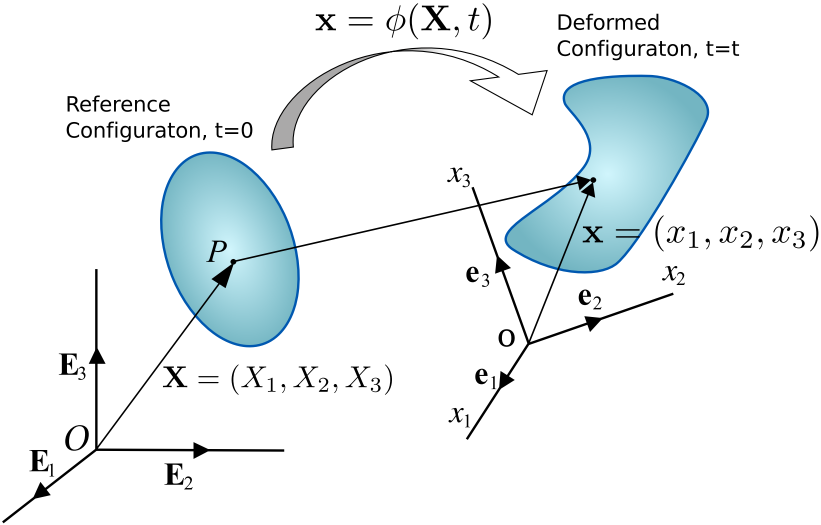
Description of Motion



(from the continuum mechanics entry, Wikipedia)

Description of Motion

- We wish to describe the generic motion of a material body ($\mathcal{B}$), including translation and rigid body rotation as well as time dependent ones.
- To trace the motion of $\mathcal{B}$, we establish an absolutely fixed (inertial) frame of reference so that points in the Euclidean space ($\mathbf{R}^3$) can be identified by their position ($\mathbf{x}$) or their coordinates (x_i, i=1,2,3).
- The subsets of $\mathbf{R}^3$ occupied by $\mathcal{B}$ are called the ***configurations*** of the body. The *initially* known configuration is particularly called *reference configuration*.

Description of Motion

- It is fundamentally important to distinguish between the particles (P) of the body and their places in $\mathbf{R}^3$: the particles should be thought of as physical entities - pieces of matter - whereas the places are merely positions in $\mathbf{R}^3$ in which particles may or may not be at any specific time.
- To identify particles, we label them in much the same way one labels discrete particles in classical dynamics. However, since $\mathcal{B}$ is a uncountable continuum of particles, we cannot use the integers to label them as in particle dynamics.

Description of Motion

- The problem is resolved by placing each particle in $\mathcal{B}$ in correspondence with an ordered triple $\mathbf{X} = (X_1, X_2, X_3)$ of real numbers. Mathematically, this "correspondence" is a *homeomorphism* from $\mathcal{B}$ into $\mathbf{R}^3$, we make no distinction between $\mathcal{B}$ and the set of particle labels.
- The numbers X_i associated with particle $\mathbf{X} \in \mathcal{B}$ are called the ***material coordinates*** of $\mathbf{X}$.

Description of Motion

- For convenience, it is customary to choose the material coordinates of $\mathbf{X}$ to exactly coincide with the ***spatial coordinates***, $\mathbf{x}$ when $\mathcal{B}$ occupies its reference configuration.
- A ***motion*** of $\mathcal{B}$ is a time-dependent family of configurations, written $\mathbf{x} = \phi(\mathbf{X}, t)$. Of course, $\mathbf{X} = \phi(\mathbf{X}, 0)$.
- To prevent weird, non-realistic behaviors, we also require configurations (i.e., the mapping ϕ) to be ***sufficiently smooth*** (to be able to take derivatives), ***invertible*** (to prevent self-penetration, for instance), and ***orientation preserving*** (to prevent a mapping to a mirror image).

Description of Motion

- ***Material velocity*** of a point $\mathbf{X}$ is defined by

$$\mathbf{V}(\mathbf{X}, t) = (\partial/\partial t)\phi(\mathbf{X}, t)$$

- Velocity viewed as a function of $(\mathbf{x}, t)$, denoted $\mathbf{v}(\mathbf{x}, t)$, is called ***spatial velocity***.

$$\mathbf{V}(\mathbf{X}, t) = \mathbf{v}(\mathbf{x}, t)$$

- ***Material acceleration*** of a motion $\phi(\mathbf{X}, t)$ is defined by

$$\mathbf{A}(\mathbf{X}, t) = \frac{\partial^2 \phi}{\partial t^2}(\mathbf{X}, t) = \frac{\partial \mathbf{V}}{\partial t}(\mathbf{X}, t)$$

By the chain rule,

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}$$

Description of Motion

- In general, if $Q(\mathbf{X}, t)$ is a material quantity–a given function of $(\mathbf{X}, t)$– and $q(\mathbf{x}, t) = Q(\mathbf{X}, t)$ is the same quantity expressed as a function of $(\mathbf{x}, t)$, then the chain rule gives

$$\frac{\partial \mathbf{Q}}{\partial t} = \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{q}.$$

- The right-hand side is called the ***material time derivative*** of a spatial field, q, and is denoted $Dq/Dt = \dot{q}$.
- Dq/Dt is the derivative of q with respect to t, holding $\mathbf{X}$ fixed, while $\partial q/\partial t$ is the derivative of q with respect to t holding $\mathbf{x}$ fixed. In particular

$$\dot{\mathbf{v}} = D\mathbf{v}/Dt = \partial \mathbf{V}/\partial t.$$

Example

$$\mathbf{x} = \phi(\mathbf{X}, t) = \left(X_1(1+t^2), X_2(1+t^2), X_3(1+t^2)\right)$$

$$\mathbf{X} = \phi^{-1}(\mathbf{x}, t) = \left(\frac{x_1}{1+t^2}, \frac{x_2}{1+t^2}, \frac{x_3}{1+t^2}\right)$$

$$\mathbf{V} = \frac{\partial \phi}{\partial t} = (2X_1 t, 2X_2 t, 2X_3 t)$$

$$\mathbf{v} = \mathbf{V}(\phi^{-1}(\mathbf{x}, t), t) = \left(\frac{2x_1 t}{1+t^2}, \frac{2x_2 t}{1+t^2}, \frac{2x_3 t}{1+t^2}\right)$$

$$\mathbf{A} = \frac{\partial \boldsymbol{V}}{\partial t} = (2X_1, 2X_2, 2X_3) \stackrel{?}{=} \frac{\partial \mathbf{v}}{\partial t}$$

$$\dot{\mathbf{v}} = \frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = ?$$

Description of Motion

- ***Deformation gradient***: The 3×3 matrix of partial derivatives of ϕ, denoted $\mathbf{F}$ and given as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$$

- Some trivial cases:
  If $\mathbf{x} = \mathbf{X}$, $\mathbf{F} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix;
  if $\mathbf{x} = \mathbf{X} + ct\mathbf{E}_1$ (translation along x-axis with speed c), $\mathbf{F} = \mathbf{I}$. Consistent with the intuition that a simple translation is not a "deformation" of the usual sense.

Description of Motion

- ***Polar decomposition***: From linear algebra, we know we can uniquely decompose $\mathbf{F}$ as

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R},$$

  where $\mathbf{R}$ is a proper orthogonal matrix called the *rotation*, and $\mathbf{U}$ and $\mathbf{V}$ are positive-definite and symmetric and called right and left *stretch tensors*[1].

- $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$ and $\mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T}$. Furthermore, we call $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ the *right Cauchy-Green tensor* and $\mathbf{b} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2$ is the *left Cauchy-Green tensor*.

[1] We didn't rigorously define tensors but all the tensors we will encounter are rank 2 and thus treated as square matrices.

Description of Motion

- ***Material displacement*** is denoted $\mathbf{U}$ and defined as

$$\mathbf{U}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$$

- ***Spatial displacement*** is denoted $\mathbf{u}$[2] and defined as

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \mathbf{X}(\mathbf{x}, t)$$

- Since $\mathbf{x} = \mathbf{U} + \mathbf{X}$, $\mathbf{F} = (\mathbf{I} + \partial\mathbf{U}/\partial\mathbf{X})$.

- Then, $\mathbf{C}$, the right Cauchy-Green tensor, becomes

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{I} + \frac{\partial\mathbf{U}}{\partial\mathbf{X}} + \left(\frac{\partial\mathbf{U}}{\partial\mathbf{X}}\right)^T + \left(\frac{\partial\mathbf{U}}{\partial\mathbf{X}}\right)^T \frac{\partial\mathbf{U}}{\partial\mathbf{X}}$$

  Note that the rotational part is not involved according to this definition. So, $\mathbf{C}$ is all about stretches.

- Green's (material or Lagrangian) strain tensor ("deviation from the unity"):

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$

[2]Note that $\mathbf{U}(\mathbf{X}, t) = \mathbf{u}(\mathbf{x}, t)$.

Description of Motion

- The spatial counterpart of **E** can be acquired through similar consideration or by "push-forwarding"[3] **E**.
- With further linearization, i.e., dropping the quadratic term under the assumption of infinitely small displacements, we get the familiar form of the spatial strain tensor (ε):

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\right] \text{ or } \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

- Also note that the following decomposition is always possible:

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{1}{2}\left[\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\right] + \frac{1}{2}\left[\frac{\partial \mathbf{u}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\right]$$

The second term represents "(rigid body) rotation".

[3] meaning the transformations from material quantities to spatial ones. See p.82 of Holzapfel (2000)

Example

$$\mathbf{x} = \phi(\mathbf{X}, t) = \left(X_1(1+t^2), X_2(1+t^2), X_3(1+t^2)\right)$$

$$\mathbf{X} = \phi^{-1}(\mathbf{x}, t) = \left(\frac{x_1}{1+t^2}, \frac{x_2}{1+t^2}, \frac{x_3}{1+t^2}\right)$$

$$\mathbf{U},\ \mathbf{u},\ \mathbf{F} = ?$$

$$\mathbf{C},\ \mathbf{E},\ \varepsilon = ?$$

Stress

- We learned how to quantitatively describe the motion of a continuum body including its “internal deformation”, which is represented by ***strain***.
- We now turn to what is the force associated with the internal deformation and how to incorporate it into the equation of force balance.
- A motion of a body is caused by two kinds of forces: **Body** and **surface** (or contact) force.
  - Gravity governing the free fall of a billiard ball: pure body force.
  - Momentum transfer by collision with another billiard ball: (mostly) surface force.
  - Easy to find examples of deformation of continua by surface forces.

Stress

- Let's consider a continuous body that is being strained by both body and surface forces.
- We need a quantity that represents the force arising due to the internal deformation.
- Such a force should be additive to the body force: i.e., the surface force is also a vector.
- Force "density":
  ex) The total graviational force is given by the integration of its density:
  $$M\mathbf{g} = \int_V \rho \mathbf{g} dV,$$
  where $\rho\mathbf{g}$ is the density.

Stress

- Generally, a body force, $\mathbf{F}_b$ is the volume integration of its density, $\mathbf{b}$:

$$\mathbf{F}_b = \int_V \mathbf{b}\, dV.$$

- Likewise, the surface force ($\mathbf{F}_s$) can also be acquired by integrating its surface density:

$$\mathbf{F}_s = \int_A \mathbf{t}\, dA.$$

  We call $\mathbf{t}$, the surface force per area, ***traction***.

- Note that we do NOT have to identify the integration surface with the physical boundary of the body.

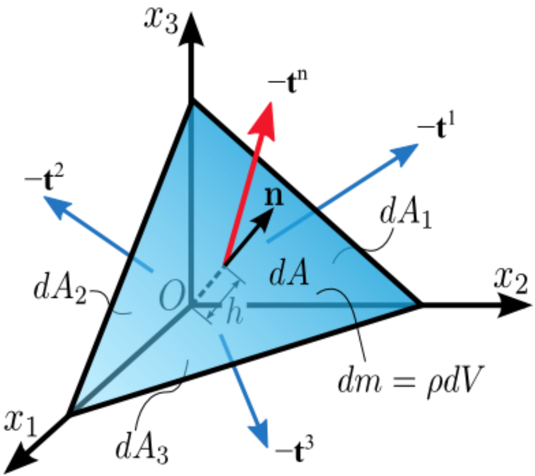
- With these force densities, we can talk about the local forces acting on a point in the body rather than on the whole body.

Stress

- Note that different tractions arise on differently oriented area even if the "state" of the material is unchanged.
- In particular, the traction is a *linear* function of the normal vector, implying the existence of a linear mapping from a normal vector to a traction vector. Since a rank 2 tensor can represent such a linear mapping, this relationship hints the idea of stress *tensor*.
- Let's look at the reasoning leading to the concept of stress tensor more carefully.

Stress

- Cauchy's tetrahedron:



Stress

- When no body force is acting, the force equilibrium states

$$\mathbf{t}^n \, dA - \mathbf{t}^1 \, dA_1 - \mathbf{t}^2 \, dA_2 - \mathbf{t}^3 \, dA_3 = \rho \left(\frac{h}{3} \, dA \right) \mathbf{a} \qquad (1)$$

- Since dA_i, i=1, . . . ,3 is projection of dA,

$$\begin{aligned} dA_1 &= \mathbf{n} dA \cdot \mathbf{e}_1 \\ dA_2 &= \mathbf{n} dA \cdot \mathbf{e}_2 \\ dA_3 &= \mathbf{n} dA \cdot \mathbf{e}_3 \end{aligned} \qquad (2)$$

- Substituting (2) into (1), we get

$$\mathbf{t}^n - \mathbf{t}^1(\mathbf{n} \cdot \mathbf{e}_1) - \mathbf{t}^2(\mathbf{n} \cdot \mathbf{e}_2) - \mathbf{t}^3(\mathbf{n} \cdot \mathbf{e}_3) = \rho \left(\frac{h}{3} \right) \mathbf{a} \qquad (3)$$

Note that dA has been cancelled out.

Stress

- In the limit $h \to 0$ and with **a** being finite, the right hand side becomes zero. Therefore,

$$\mathbf{t}^n = \mathbf{t}^1 \; n_1 + \mathbf{t}^2 \; n_2 + \mathbf{t}^3 \; n_3 \tag{4}$$

- Eq. (4) further implies that there is a rank 2 tensor[4], $\boldsymbol{\sigma}$, such that

$$\mathbf{t}^n = \boldsymbol{\sigma} \; \mathbf{n}, \tag{5}$$

  where the column vectors of $\boldsymbol{\sigma}$ are $\mathbf{t}^i$ $(i = 1 \ldots 3)$.

- We call the rank 2 tensor $\boldsymbol{\sigma}$ the ***Cauchy stress tensor***.

- Note that all the considerations so far have been made with respect to the *current* (or deformed) configuration.

[4] Again, we identify rank 2 tensors with square matrices

Properties of Cauchy Stress Tensor

- Cauchy stress, σ, is and gotta be ***symmetric*** to be physically meaningful. For a proof, wait until we get to the principle of angular momentum balance.
- We make frequent use of ***invariants***, ***principal stresses*** and associated directions.

Balance Laws: Transport Theorem

- Reminder: material time derivative
  - Time derivative of a quantity Q in the referece configuration:

$$\frac{\partial Q}{\partial t} = \frac{DQ}{Dt}.$$

  - Time derivative of the same quantity expressed in the current configuration: Since $q(\mathbf{x}(\mathbf{X}, t), t) = Q(\mathbf{X}, t)$,

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \frac{\partial \mathbf{x}}{\partial t} \cdot \frac{\partial q}{\partial \mathbf{x}} = \frac{\partial q}{\partial t} + (\mathbf{v} \cdot \nabla)q. \tag{6}$$

- Reynold's transport theorem

$$\begin{aligned} \frac{D}{Dt} \int_{v(t)} f \, dv &= \int_{v(t)} \left(\frac{Df}{Dt} + f \nabla \cdot \mathbf{v} \right) dv \\ &= \int_{v(t)} \left(\frac{\partial f}{\partial t} + \nabla \cdot (f\mathbf{v}) \right) dv \end{aligned} \tag{7}$$

Balance Laws: Transport Theorem

- Proof of the transport theorem:
  By change of variables, and differentiating under the integral sign,

$$\frac{D}{Dt}\int_{v(t)} f dv = \frac{D}{Dt}\int_V f(\phi(\mathbf{X},t),t)J(\mathbf{X},t)dV$$
$$= \int_V \left[\left(\frac{D}{Dt}f(\phi(\mathbf{X},t),t)\right)J(\mathbf{X},t) + f(\phi(\mathbf{X},t),t)\left(\frac{D}{Dt}J(\mathbf{X},t)\right)\right]dV.$$

  $DJ/Dt = (\nabla\cdot\mathbf{v})J$, where J is $\det(\partial\mathbf{x}/\partial\mathbf{X})$. Inserting this in the preceding expression and changing variables back to $\mathbf{x}$ gives the result.

- The physical meaning is that the rate of change of a quantity contained within the current configuration is equal to the time rate of change of the quantity and its net flux associated with the motion of material.

Balance Laws: Mass balance (or conservation)

- Mass conservation (material is neither created nor lost during deformation):

$$\frac{D}{Dt}\int_V R\,dV = \frac{D}{Dt}\int_{v(t)} \rho(\mathbf{x}, t)dv = 0 \tag{8}$$

- By Reynold's transport theorem, we get

$$\int_{v(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right] dv = 0. \tag{9}$$

- Since Eq. (9) should hold for arbitrary subset of the body, the integrand itself must vanish everywhere. Therefore, we get the usual form of the mass conservation equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{10}$$

Balance Laws: Force Balance (or conservation of linear momentum)

- For a mass particle:

$$\frac{d\mathbf{p}}{dt} = \sum_i \mathbf{F}_i, \tag{11}$$

  where $\mathbf{p}$ is the linear momentum of the particle and $\mathbf{F}_i$ is the i-th force acting on it.

- For a continuous body,

$$\frac{D}{Dt}\int_{v(t)} \rho\,\mathbf{v}\,dv = \int_{v(t)} \mathbf{b}\,dv + \int_{\partial v(t)} \mathbf{t}\,dS \tag{12}$$

  or if we introduce Cauchy stress into the above equation, we get

$$\frac{D}{Dt}\int_{v(t)} \rho\,\mathbf{v}\,dv = \int_{v(t)} \mathbf{b}\,dv + \int_{\partial v(t)} \boldsymbol{\sigma}\mathbf{n}\,dS \tag{13}$$

Balance Laws: Force Balance (or conservation of linear momentum)

- By the Gauss's theorem,

$$\int_{\partial v(t)} \boldsymbol{\sigma}\mathbf{n}\, dS = \int_{v(t)} \nabla \cdot \boldsymbol{\sigma}\, dv. \tag{14}$$

- Therefore, (13) becomes

$$\int_{v(t)} (\nabla \cdot \boldsymbol{\sigma} + \rho\, \mathbf{b})\, dv - \frac{D}{Dt}\int_{v(t)} \rho\, \mathbf{v}\, dv = 0. \tag{15}$$

- We apply Reynold's transport theorem to the second term on the left hand side of Eq. (15). Interestingly, we get the following identity:

$$\frac{D}{Dt}\int_{v(t)} \rho\, \mathbf{v}\, dv = \int_{v(t)} \rho\, \frac{D\mathbf{v}}{Dt}\, dv.$$

Balance Laws: Force Balance (or conservation of linear momentum)

- Plugging the previous identity into Eq. (15), we get

$$\int_{v(t)} \left[\nabla \cdot \boldsymbol{\sigma} + \rho\, \mathbf{b} - \rho \frac{D\mathbf{v}}{Dt} \right] dv = 0. \tag{16}$$

- Since Eq. (16) should hold not only for the entire volume but also for any arbitrary subset of the body, the integrand itself should be zero. Consequently, we obtain the *local* equation of motion or force balance:

$$\nabla \cdot \boldsymbol{\sigma} + \rho\, \mathbf{b} = \rho\, \frac{D\mathbf{v}}{Dt}. \tag{17}$$

Balance Laws: Force Balance (or conservation of linear momentum)

- If the motion is not time dependent, meaning either a static equilibrium (all the velocities are zero) or a steady state (all the velocities are constant, possibly non-zero), the inertial term of Eq. (17) is zero and the local equation of motion becomes

$$\nabla \cdot \boldsymbol{\sigma} + \rho\, \mathbf{b} = 0. \tag{18}$$

This is the most frequently encountered form in geodynamics and is called *Stoke's equation*.

Energy Balance Equation

- For simplicity, (1) we consider a body and only heat energy in it.
  - What are other energies that we could consider?
- Deformation of the body means work done to the body and/or by the body, which will lead to change in internal energy. So, (2) we do not consider deformation here.
- We further assume that (3) there is no heat energy sink or source.
- Finally, (4) we consider only heat transfer by *conduction*.

Heat Energy

- *Heat capacity at constant pressure*, C_p:

$$C_p = \left(\frac{\partial Q}{\partial T}\right)_p, \tag{19}$$

where Q is the heat energy.

- *Specific heat capacity at constant pressure*, c_p:

$$c_p = \frac{C_p}{m} \tag{20}$$

where m is mass.

- *Heat energy per mass*, q:

$$q = \int_0^T c_p(T)dT \tag{21}$$

or if c_p is not a function of temperature,

$$q = c_p T. \tag{22}$$

Heat Energy

- *Heat energy of a body*, Q:

$$Q = \int_V \rho c_p T dV. \tag{23}$$

- Under the set of assumptions listed above, the law of energy conservation states that the time rate of change of heat energy within a body is equal to the net flux of heat energy through its boundaries:

$$\frac{D}{Dt}\int_V \rho c_p T dV = \int_{\partial V} \mathbf{f}\cdot\mathbf{n} dS, \tag{24}$$

where **f** is heat flux, representing heat energy flowing through unit area per unit time.

- *Fourier's law of heat conduction*:

$$\mathbf{f} = k\nabla T, \tag{25}$$

where k is *heat conductivity*.

Heat Energy

- The energy conservation equation becomes

$$\frac{D}{Dt}\int_V \rho c_p T dV = \int_{\partial V} k\nabla T \cdot \mathbf{n} dS. \tag{26}$$

- By applying the divergence theorem to the r.h.s and bringing the time derivative into the integral on the l.h.s, we get

$$\int_V \frac{\partial}{\partial t}(\rho c_p T) dV = \int_V \nabla \cdot (k\nabla T) dV. \tag{27}$$

  Note that material time derivative is identical to partial time derivative since spatial velocity is zero.

- Let's further assume that material properties, ρ, c and k are constant.

$$\int_V \rho c_p \frac{\partial T}{\partial t} dV = \int_V k\nabla^2 T dV. \tag{28}$$

Heat Energy

- Since the energy conservation should be true for any arbitrary neighborhood around a point in the body,

$$\int_V \left(\rho c_p \frac{\partial T}{\partial t} - k \nabla^2 T \right) dV = 0 \tag{29}$$

for an arbitrary V, meaning the integrand should be identically zero.

- We finally arrive at the familiar form of the “heat equation”:

$$\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T. \tag{30}$$

- Note that the left hand side of (26) could have been more complicated according to Reynold’s transport theorem. It wasn’t because of our assumption that the body doesn’t deform.

Heat Advection-Diffusion

- We want to slightly generalize the heat diffusion equation to the heat **advection-diffusion** equation. The new equation will describe thermal energy that is not only diffused but also carried along with a deforming continuous medium.
- Note that the diffusion equation is derived from the conservation of thermal energy:

$$\frac{D}{Dt}\int_{V(t)} \rho c_p T dV = \int_{\partial V(t)} k \nabla T \cdot \mathbf{n} dS. \tag{31}$$

- We previously assumed that the continuum body in which temperature is non-uniform such that diffusion occurs is not deforming. So, the volume V in the above equation was a constant.

Heat Advection-Diffusion

- Let's remove this assumption because when the medium is in motion, the volume is also time-dependent.
- By applying the Reynold's transport theorem to the l.h.s of Eq. (31) and the divergence theorem to the r.h.s, we get

$$\int_{v(t)} \left(\frac{\partial(\rho c_p T)}{\partial t} + \mathbf{v} \cdot \nabla(\rho c_p T) \right) dv = \int_{v(t)} \nabla \cdot (k \nabla T)\, dv. \tag{32}$$

- If the continuous media is *compressible*, deformation causes pV (pressure-volume, i.e., mechanical) work, which contributes the overall thermal energetics. However, if the media is **incompressible** or can **freely expand/contract**, it does not do any mechanical work.

Heat Advection-Diffusion

- Furthermore, when the continuous medium is going through *shearing*, in general we cannot ignore shear heating as a source term. In some cases, however, we can ignore shear heating. An example can be a plate with a prescribed thickness that is translating in one direction without internal deformation.

- If there are no other heat sources/sinks to consider, the assumptions of zero pV work, zero shear heating and constant material properties give

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T. \tag{33}$$

How to Relate Stress and Strain

- We have considered deformation of continua and balance laws in them.
- These kinematics and mechanics apply to all the continuous media. Then, where do the characteristics of individual material come from?
- Properties unique to a certain material are determined by the material's internal constitution or physical make-up. The quantitative expressions for such internal constitution are called ***constitutive equations / laws / relations / models***.

Linear Elasticity

- Hooke's law for a 1 dimensional mass-spring system:

$$F = -kx$$

- If no damping force acts on it, the system is *conservative*, meaning by definition that there is a potential function $U(x)$ such that $F = -\nabla U$.
- In this 1D example, integration to get U is straightforward and $U = \frac{1}{2}kx^2$.

Linear Elasticity

- A material is called ***ideally elastic*** when a body formed of the material recovers its original form completely upon removal of the forces causing the deformation, and there is a one-to-one relationship between the state of stress and the state of strain, for a given temperature.
- The one-to-one relationship precludes behaviors like creep at constant load or stress relaxation at constant strain.
- The classical elastic constitutive equations, often called the ***generalized Hooke's law***, are nine equations expressing the stress components as linear homonenous (i.e., all the terms are of the same power) functions of the nine strain components:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{34}$$

Linear Elasticity

- The rank 4 tensor, C_{ijkl}, has $81 (= 3^4)$ components.
- However, recall that stress and strain tensor are symmetric: i.e., $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{kl} = \varepsilon_{lk}$.
- Thus,
$$C_{ijkl} = C_{jikl} \text{ and } C_{ijkl} = C_{ijlk}.$$
- We further consider the case in which the material is ***elastically isotropic***, i.e., there are no preferred directions in the material. Then, the elastic constants (C_{ijkl}) must be the same at a given particle for all possible choices of rectangular Cartesian coordinates in which stress and strain components are evaluated.

Linear Elasticity

- The most general rank 4 tensor that satisfy all of the above symmetry and isotropy conditions is

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \tag{35}$$

(see Malvern Sec. 6.1 and 6.2 for further details.)

- The constitutive relation becomes

$$\sigma_{ij} = \left[\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right] \varepsilon_{kl} \tag{36}$$

- Finally, after some simplification, we reach the isotropic generalized Hooke's law:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}, \tag{37}$$

where λ and μ are called Lamé's constants.

The full set of governing equations

- Mass conservation: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$
- (Linear-)Momentum conservation: $\nabla \cdot \boldsymbol{\sigma} + \rho\, \mathbf{b} = \rho\, \frac{D\mathbf{v}}{Dt}.$
- Energy conservation: $\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T.$
- Constitutive law: $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\epsilon, \dot{\epsilon}, T, p, etc).$